

Dynamic network models and graphon estimation

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Abstract

In the present paper we consider a dynamic stochastic network model. The objective is estimation of the tensor of connection probabilities $\mathbf{\Lambda}$ when it is generated by a Dynamic Stochastic Block Model (DSBM) or a dynamic graphon. In particular, in the context of DSBM, we derive penalized least squares estimator $\hat{\mathbf{\Lambda}}$ of $\mathbf{\Lambda}$ and show that $\hat{\mathbf{\Lambda}}$ satisfies an oracle inequality and also attains minimax lower bounds for the risk. We extend those results to estimation of $\mathbf{\Lambda}$ when it is generated by a dynamic graphon function. The estimators constructed in the paper are adaptive to the unknown number of blocks in the context of DSBM or of the smoothness of the graphon function. The technique relies on the vectorization of the model and leads to much simpler mathematical arguments than the ones used previously in the stationary set up. In addition, all our results are non-asymptotic and allow a variety of extensions.

Keywords and phrases: dynamic network, graphon, stochastic block model, nonparametric regression, minimax rate

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1 Introduction

Networks arise in many areas of research such as sociology, biology, genetics, ecology, information technology to list a few. An overview of statistical modeling of random graphs can be found in, e.g., Kolaczyk (2009) and Goldenberg *et al.* (2011). While static network models are relatively well understood, the literature on the dynamic network models is fairly recent.

In this paper, we consider a dynamic network defined as an undirected graph with n nodes with connection probabilities changing in time. Assume that we observe the values of a tensor $\mathbf{B}_{i,j,l} \in \{0,1\}$ at the time t_l where $0 < t_1 < \dots < t_L = T$. For simplicity, we assume that time instances are equispaced and the time interval is scaled to one, i.e. $t_l = l/L$. Here $\mathbf{B}_{i,j,l} = 1$ if a connection between nodes i and j is observed and $\mathbf{B}_{i,j,l} = 0$ otherwise. We set $\mathbf{B}_{i,i,l} = 0$ and $\mathbf{B}_{i,j,l} = \mathbf{B}_{j,i,l}$ for any $i, j = 1, \dots, n$ and $l = 1, \dots, L$, and assume that $\mathbf{B}_{i,j,l}$ are independent Bernoulli random variables with $\mathbf{\Lambda}_{i,j,l} = \mathbb{P}(\mathbf{B}_{i,j,l} = 1)$ and $\mathbf{\Lambda}_{i,i,l} = 0$.

In this paper, we consider a Dynamic Stochastic Block Model (DSBM) where all n nodes are grouped into m classes $\Omega_1, \dots, \Omega_m$, and probability of a connection $\mathbf{\Lambda}_{i,j,l}$ is entirely determined by the groups to which the nodes i and j belong at the moment t_l . In particular, if $i \in \Omega_k$ and $j \in \Omega_{k'}$, then $\mathbf{\Lambda}_{i,j,l} = \mathbf{G}_{k,k',l}$. Here, \mathbf{G} is the *connectivity tensor* at time t_l with $\mathbf{G}_{k,k',l} = \mathbf{G}_{k',k,l}$.

The dynamic network described above can be generalized as a dynamic graphon. Let $\boldsymbol{\zeta} = (\zeta_1, \dots, \zeta_n)$ be a random vector sampled from a distribution $\mathbb{P}_{\boldsymbol{\zeta}}$ supported on $[0,1]^n$. Although the most common choice for $\mathbb{P}_{\boldsymbol{\zeta}}$ is the i.i.d. uniform distribution for each ζ_i , we do not make this assumption in the present paper. We further assume that there exists a function $f : [0,1]^3 \rightarrow [0,1]$ such that for any t one has $f(x,y,t) = f(y,x,t)$ and

$$\mathbf{\Lambda}_{i,j,l} = f(\zeta_i, \zeta_j, t_l), \quad i, j = 1, \dots, n, \quad l = 1, \dots, L. \quad (1.1)$$

Then, function f summarizes behavior of the network and can be called a *dynamic graphon*, similarly to the situation of a stationary network. This formulation allows to study a different set of stochastic network models than the DSBM. It is known that graphons play an important role in the theory of graph limits described in Lovász and Szegedy (2006) and Lovász (2012).

Given an observed adjacency tensor \mathbf{B} sampled according to model (1.1), the graphon function f is not identifiable since the topology of a network is invariant with respect to any change of labeling of its nodes. Therefore, for any f and any measure-preserving bijection $\mu : [0, 1] \rightarrow [0, 1]$ (with respect to Lebesgue measure), the functions $f(x, y, t)$ and $f(\mu(x), \mu(y), t)$ define the same probability distribution on random graphs. For this reason, we are considering equivalence classes of graphons. Note that in order it is possible to compare clustering of nodes across time instances, we introduce an assumption (which, to the best of our knowledge, first appeared in Matias and Miele (2015)) that there are no label switching in time, that is, every node carries the same label at any time t_l , so that function μ is independent of t .

The objective of the paper is estimation of the tensor \mathbf{A} from observations \mathbf{B} in the case of DSBM and the dynamic graphon (1.1). It is reasonable to assume that the tensor \mathbf{G} does not change significantly with time. The nodes of the network, however, can switch between groups and this can significantly affect the tensor \mathbf{A} .

In the last few years, dynamic network models attracted a great deal of attention (see, e.g., Durante *et al.* (2015), Durante *et al.* (2016), Han *et al.* (2015), Kolar *et al.* (2010), Matias and Miele (2015), Minhas *et al.* (2015), Xing *et al.* (2010), Xu (2015), Xu and Hero III (2014) and Yang *et al.* (2011) among others). Majority of those paper describe changes in the connection probabilities and group memberships via various kinds of Bayesian or Markov random field models and carry out the inference using the EM or iterative optimization algorithms. While procedures described in these papers show good computational properties, they come without guarantees for the estimation precision.

On the other hand, very recently, several authors carried out minimax studies in the context of stationary network models. In particular, Gao *et al.* (2015) developed the upper and the minimax lower bounds for the risk of estimation of the matrix of connection probabilities. Klopp *et al.* (2016) extended these results to the case when the network is sparse in a sense that probability of connection is uniformly small and tends to zero as $n \rightarrow \infty$. Also, Zhang and Zhou (2015) investigated minimax rates of community detection in two-class stochastic block model. Our paper can be viewed as a nontrivial extension of the above results to the case of time-dependent network models and graphons. To the best of our knowledge, this is the first minimax study of dynamic networks and graphons: the only paper known to us that is concerned with estimation precision in dynamic network models is by Han *et al.* (2015) where the authors are studying consistency of their procedures when $n \rightarrow \infty$ or $L \rightarrow \infty$.

In the present paper, under the assumption that tensor \mathbf{G} is smooth as a function of time, in Section 3, we derive penalized least squares estimators $\hat{\mathbf{A}}$ of \mathbf{A} and show that they satisfy oracle inequalities. The technique relies on the vectorization of the model, described in Section 2.2, and allows one to take advantage of the well studied methodologies in nonparametric regression estimation. Initially, we do not place any assumptions on the changes in the memberships of the nodes in time, however, in Section 3, as a particular case, we consider a situation where only at most n_0 nodes can change their memberships between two consecutive time points. For the DSBM, under the latter assumption, in Section 4, we derive minimax lower bounds for the risk of any estimator of \mathbf{A} and confirm that the estimators constructed in the paper attain those lower bounds. In Section 5 we extend those results to estimation of the tensor \mathbf{A} when it is generated by a graphon function. We show that the estimators are minimax optimal within a logarithmic factor of L . Estimators,

constructed in the paper, do not require knowledge of the number of classes m in the context of the DSBM, or a degree of smoothness of the graphon function f if $\mathbf{\Lambda}$ is generated by a dynamic graphon. Finally, in Section 6 we discuss various generalizations of the technique proposed in the paper. For example, it can be adapted to a situation where the number of nodes in the network depends on time or when the connection probabilities have jump discontinuities. Moreover, since our oracle inequality for the DSBM is derived under no assumptions about clustering mechanism, one can accommodate a much more diverse collection of scenarios than the ones studied in the paper. The proofs of all statements are placed into Section 7.

Note that unlike in Klopp *et al.* (2016) we do not consider a network that is sparse in a sense that probabilities of connections between classes are uniformly small. However, since our technique is based on model selection, it allows to study a network where probabilities of some connections are equal to zero as it is done in Bickel and Chen (2009), Bickel *et al.* (2011) or Wolfe and Olhede (2013). In particular, from our theory, one automatically obtains an oracle inequality for a sparse stationary stochastic block model (SBM).

The present paper makes several key contributions. First, to the best of our knowledge, this is the first minimax study of estimation of the tensor of connection probabilities in a dynamic stochastic network model. Second, we provide estimators that are adaptive to the number of blocks in the context of the DSBM and adaptive to the smoothness of the graphon function. In the case of $L = 1$ we automatically obtain an adaptive estimator in a sparse SBM model where probabilities of interactions between some groups of nodes are identically equal to zero. All our results are non-asymptotic. Finally, for derivation of estimators and their analysis, we use a novel approach that is based on vectorization of the model and leads to much simpler mathematical arguments than the ones used in both Gao *et al.* (2015) and Klopp *et al.* (2016). The latter feature may be generally useful for research in the area of network analysis. Finally, the technique allows various extensions that we discuss later in Section 6.

2 Notation and data structures

2.1 Notation

For any two positive sequences $\{a_n\}$ and $\{b_n\}$, $a_n \asymp b_n$ means that there exists a constant $C > 0$ independent of n such that $C^{-1}a_n \leq b_n \leq Ca_n$ for any n . For any set Ω , denote cardinality of Ω by $|\Omega|$. For any x , $[x]$ is the largest integer no larger than x .

For any vector $\mathbf{t} \in \mathbb{R}^p$, denote its ℓ_2 , ℓ_1 , ℓ_0 and ℓ_∞ norms by, respectively, $\|\mathbf{t}\|$, $\|\mathbf{t}\|_1$, $\|\mathbf{t}\|_0$ and $\|\mathbf{t}\|_\infty$. Denote by $\|\mathbf{t}_1 - \mathbf{t}_2\|_H$ the Hamming distance between vectors \mathbf{t}_1 and \mathbf{t}_2 . Denote by $\mathbf{1}$ and $\mathbf{0}$ vectors that have, respectively, only unit or zero elements. Denote by \mathbf{e}_j the vector with 1 in the j -th position and all other elements equal to zero.

For a matrix \mathbf{A} , its i -th row and j -th columns are denoted, respectively, by $\mathbf{A}_{i,*}$ and $\mathbf{A}_{*,j}$. Similarly, for a tensor $\mathbf{A} \in \mathbb{R}^{n_1 \times n_2 \times n_3}$, we denote its l -th $(n_1 \times n_2)$ -dimensional sub-matrix by $\mathbf{A}_{*,*,l}$. Let $\text{vec}(\mathbf{A})$ be the vector obtained from matrix \mathbf{A} by sequentially stacking its columns. Denote by $\mathbf{A} \otimes \mathbf{B}$ the Kronecker product of matrices \mathbf{A} and \mathbf{B} . Also, \mathbf{I}_k is the identity matrix of size k . For any subset J of indices, any vector \mathbf{t} and any matrix \mathbf{A} , denote the restriction of \mathbf{t} to indices in J by \mathbf{t}_J and the restriction of \mathbf{A} to columns $\mathbf{A}_{*,j}$ with $j \in J$ by \mathbf{A}_J . Also, denote by $\mathbf{t}_{(J)}$ the modification of vector \mathbf{t} where all elements \mathbf{t}_j with $j \notin J$ are set to zero.

For any matrix \mathbf{A} , denote its spectral and Frobenius norms by, respectively, $\|\mathbf{A}\|$ and $\|\mathbf{A}\|_F$. Denote $\|\mathbf{A}\|_H \equiv \|\text{vec}(\mathbf{A})\|_H$ and $\|\mathbf{A}\|_0 \equiv \|\text{vec}(\mathbf{A})\|_0$. For any tensor $\mathbf{A} \in \mathbb{R}^{n_1 \times n_2 \times n_3}$, denote $\|\mathbf{A}\|_2^2 = \sum_{l=1}^{n_3} \|\mathbf{A}_{*,*,l}\|_F^2$.

Denote by $\mathcal{M}(m, n)$ a collection of *membership* (or *clustering*) matrices $\mathbf{Z} \in \{0, 1\}^{n \times m}$, i.e. matrices such that \mathbf{Z} has exactly one 1 per row and $\mathbf{Z}_{ik} = 1$ iff a node i belongs to the class Ω_k and is zero otherwise. Denote by $\mathcal{C}_{n, m}$ a set of clustering matrices such that

$$\mathcal{C}(m, n, L) \subseteq \prod_{l=1}^L \mathcal{M}(m, n). \quad (2.1)$$

2.2 Vectorization of the model

Let $\mathbf{Z}^{(l)} \in \mathcal{M}(m, n)$ be the clustering matrix at the moment t_l . Then, it is easy to check that

$$\mathbf{\Lambda}_{*, *, l} = \mathbf{Z}^{(l)} \mathbf{G}_{*, *, l} (\mathbf{Z}^{(l)})^T,$$

Apply operation of vectorization to $\mathbf{\Lambda}_{*, *, l}$. Denote

$$\boldsymbol{\lambda}^{(l)} = \text{vec}(\mathbf{\Lambda}_{*, *, l}), \quad \mathbf{b}^{(l)} = \text{vec}(\mathbf{B}_{*, *, l}), \quad \mathbf{g}^{(l)} = \text{vec}(\mathbf{G}_{*, *, l}). \quad (2.2)$$

Then, Theorem 1.2.22(i) of Gupta and Nagar (2000) yields

$$\boldsymbol{\lambda}^{(l)} = (\mathbf{Z}^{(l)} \otimes \mathbf{Z}^{(l)}) \mathbf{g}^{(l)} \quad (2.3)$$

Note that

$$\mathbf{b}_i^{(l)} \sim \text{Bernoulli}(\boldsymbol{\lambda}_i^{(l)}), \quad i = 1, \dots, n^2, \quad (2.4)$$

where $\mathbf{b}_i^{(l)}$ are independent for different values of l but not i due to the symmetry. In addition, some values of $\mathbf{b}_i^{(l)}$ and $\boldsymbol{\lambda}_i^{(l)}$ are equal to zero by construction. In order to remove redundant entries from $\boldsymbol{\lambda}^{(l)}$ and $\mathbf{b}^{(l)}$, we use double-indexing. In particular, we replace the index i in $\boldsymbol{\lambda}^{(l)}$ corresponding to the element $\mathbf{\Lambda}_{i_1, i_2, l}$ by $\mathbf{i} = (i_1, i_2)$. Remove all rows \mathbf{i} in $(\mathbf{Z}^{(l)} \otimes \mathbf{Z}^{(l)})$ and $\boldsymbol{\lambda}^{(l)}$ that correspond to $i_1 \geq i_2$. Then, (2.3) and (2.4) can be re-written as

$$\boldsymbol{\theta}^{(l)} = \tilde{\mathbf{C}}^{(l)} \mathbf{g}^{(l)}, \quad \mathbf{a}_i^{(l)} \sim \text{Bernoulli}(\boldsymbol{\theta}_i^{(l)}), \quad i = 1, \dots, n(n-1)/2, \quad (2.5)$$

where $\boldsymbol{\theta}^{(l)}$ and $\mathbf{a}^{(l)}$ are the reductions of vectors $\boldsymbol{\lambda}^{(l)}$ and $\mathbf{b}^{(l)}$ to the collection of indices $\mathbf{i} = (i_1, i_2)$ with $1 \leq i_1 < i_2 \leq n$ and matrix $\tilde{\mathbf{C}}^{(l)}$ is obtained from $(\mathbf{Z}^{(l)} \otimes \mathbf{Z}^{(l)})$ by removing the rows with $i_1 \geq i_2$. Note that unlike $\mathbf{b}^{(l)}$, elements $\mathbf{a}_i^{(l)}$ and $\mathbf{a}_{i'}^{(l)}$ are independent if $i \neq i'$ or $l \neq l'$.

Observe that although we removed the redundant elements from vectors $\boldsymbol{\lambda}^{(l)}$ and $\mathbf{b}^{(l)}$, we have not done so for the vector $\mathbf{g}^{(l)}$. Indeed, if we replace the index k in $\mathbf{g}_k^{(l)}$ corresponding to the element $\mathbf{G}_{k_1, k_2, l}$ by $\mathbf{k} = (k_1, k_2)$, then (2.5) implies that

$$\begin{aligned} \boldsymbol{\theta}_{i_1, i_2}^{(l)} &= \sum_{k_1=1}^m \sum_{k_2=1}^m \tilde{\mathbf{C}}_{i_1, i_2, k_1, k_2}^{(l)} \mathbf{g}_{k_1, k_2}^{(l)} \\ &= \sum_{k_1=1}^m \sum_{k_2=k_1}^m [\tilde{\mathbf{C}}_{i_1, i_2, k_1, k_2}^{(l)} + \tilde{\mathbf{C}}_{i_1, i_2, k_2, k_1}^{(l)} \mathbb{I}(k_1 \neq k_2)] \mathbf{g}_{k_1, k_2}^{(l)}. \end{aligned}$$

Denote the reduction of the vector $\mathbf{g}^{(l)}$ to components $\mathbf{k} = (k_1, k_2)$ with $1 \leq k_1 \leq k_2 \leq m$ by $\mathbf{q}^{(l)}$ and

$$\mathbf{C}_{i_1, i_2, k_1, k_2}^{(l)} = \tilde{\mathbf{C}}_{i_1, i_2, k_1, k_2}^{(l)} + \tilde{\mathbf{C}}_{i_1, i_2, k_2, k_1}^{(l)} \mathbb{I}(k_1 \neq k_2), \quad 1 \leq i_1 < i_2 \leq n, \quad 1 \leq k_1 \leq k_2 \leq m. \quad (2.6)$$

Let us reflect on the structure of the matrix $\mathbf{C}^{(l)}$. Recall that matrix $\tilde{\mathbf{C}}^{(l)}$ was obtained from the matrix $(\mathbf{Z}^{(l)} \otimes \mathbf{Z}^{(l)})$ by removing some of the rows. Therefore, using double-indexing, we obtain that $\tilde{\mathbf{C}}_{i_1 i_2 k_2 k_1}^{(l)} = 1$ only if $i_1 \in \Omega_{k_1}$ and $i_2 \in \Omega_{k_2}$, so that it has exactly one 1 per row. Matrix $\mathbf{C}^{(l)}$ is obtained from matrix $\tilde{\mathbf{C}}^{(l)}$ by adding the columns corresponding to indices (k_1, k_2) and (k_2, k_1) with $k_1 \neq k_2$. Therefore, matrix $\mathbf{C}^{(l)}$ is also a binary matrix with exactly one 1 per row and, hence, it is a clustering matrix: $\mathbf{C}_{i_1 i_2 k_1 k_2}^{(l)} = 1$ iff $i_1 \in \Omega_{k_1}$ and $i_2 \in \Omega_{k_2}$ or $i_1 \in \Omega_{k_2}$ and $i_2 \in \Omega_{k_1}$ where $1 \leq i_1 < i_2 \leq n$ and $1 \leq k_1 \leq k_2 \leq m$. Hence, $\mathbf{C}^{(l)} \in \mathcal{M}(M, N)$ and one has

$$\mathbf{a}^{(l)} = \boldsymbol{\theta}^{(l)} + \boldsymbol{\xi}^{(l)} \quad \text{with} \quad \boldsymbol{\theta}^{(l)} = \mathbf{C}^{(l)} \mathbf{q}^{(l)}, \quad l = 1, \dots, L, \quad (2.7)$$

where $\boldsymbol{\theta}^{(l)} \in \mathbb{R}^N$, $\mathbf{q}^{(l)} \in \mathbb{R}^M$, $N = n(n-1)/2$ and $M = m(m+1)/2$. Here, components $\mathbf{a}_i^{(l)}$ of vector $\mathbf{a}^{(l)}$ are independent Bernoulli variables with $\mathbb{P}(\mathbf{a}_i^{(l)} = 1) = \theta_i^{(l)}$, so that components of vectors $\boldsymbol{\xi}^{(l)}$ are independent for different values of i or l .

Consider matrices $\mathbf{A}, \boldsymbol{\Theta} \in \mathbb{R}^{N \times L}$ and $\mathbf{Q} \in \mathbb{R}^{M \times L}$ with columns $\mathbf{a}^{(l)}$, $\boldsymbol{\theta}^{(l)}$ and $\mathbf{q}^{(l)}$, respectively, and denote $\mathbf{a} = \text{vec}(\mathbf{A})$, $\boldsymbol{\theta} = \text{vec}(\boldsymbol{\Theta})$ and $\mathbf{q} = \text{vec}(\mathbf{Q})$. Note that vectors $\mathbf{a}, \boldsymbol{\theta} \in \mathbb{R}^{NL}$ and $\mathbf{q} \in \mathbb{R}^{ML}$ are obtained by stacking vectors $\mathbf{a}^{(l)}$, $\boldsymbol{\theta}^{(l)}$ and $\mathbf{q}^{(l)}$ vertically for $l = 1, \dots, L$. Define a block diagonal matrix $\mathbf{C} \in \{0, 1\}^{NL \times ML}$ with blocks $\mathbf{C}^{(l)}$, $l = 1, \dots, L$, on the diagonal. Then, (2.7) implies that

$$\mathbf{a} = \boldsymbol{\theta} + \boldsymbol{\xi} \quad \text{with} \quad \boldsymbol{\theta} = \mathbf{C} \mathbf{q} = \mathbf{C} \text{vec}(\mathbf{Q}), \quad (2.8)$$

where \mathbf{a}_i are independent Bernoulli(θ_i) variables, $i = 1, \dots, NL$.

Observe that if the matrix \mathbf{C} were known, then equations in (2.8) would represent a regression model with independent Bernoulli errors. Moreover, note that, for every l , matrix $(\mathbf{C}^{(l)})^T \mathbf{C}^{(l)} = (\mathbf{D}^{(l)})^2$ is diagonal with $\mathbf{D}_{k_1, k_2}^{(l)} = \sqrt{N_{k_1, k_2}}$, where N_{k_1, k_2} is the number of pairs (i_1, i_2) of nodes such that $i_1 < i_2$ and one node is in class Ω_{k_1} while another is in class Ω_{k_2} .

3 Assumptions and oracle inequalities for the DSBM

It is reasonable to assume that the values of the probabilities $\mathbf{q}^{(l)}$ of connections do not change dramatically from one time instant to another. In other words, we assume that, for various $k = 1, \dots, M$, vectors $\mathbf{q}_k = (\mathbf{q}_k^{(1)}, \dots, \mathbf{q}_k^{(L)})$ represent values of smooth functions. In order to quantify this phenomenon, we assume that vectors \mathbf{q}_k have sparse representation in some orthogonal basis $\mathbf{H} \in \mathbb{R}^{L \times L}$ with $\mathbf{H}^T \mathbf{H} = \mathbf{H} \mathbf{H}^T = \mathbf{I}_L$, so that vector $\mathbf{H} \mathbf{q}_k^T$ has only few nonzero coefficients. Using matrix \mathbf{Q} defined in the previous section, the latter can be expressed as $\|\mathbf{H} \mathbf{Q}^T\|_0 = \|\mathbf{Q} \mathbf{H}^T\|_0$ being small. Note that by Theorem 1.2.22 of Gupta and Nagar (2000), one has

$$\text{vec}(\mathbf{Q} \mathbf{H}^T) = (\mathbf{H} \otimes \mathbf{I}_M) \text{vec}(\mathbf{Q}) = (\mathbf{H} \otimes \mathbf{I}_M) \mathbf{q}, \quad (3.1)$$

where $\mathbf{W} = (\mathbf{H} \otimes \mathbf{I}_M)$ is an orthogonal matrix such that $\mathbf{W}^T \mathbf{W} = \mathbf{W} \mathbf{W}^T = \mathbf{I}_{ML}$. Denote

$$\mathbf{d} = \mathbf{W} \mathbf{q}, \quad \mathbf{d} \in \mathbb{R}^{ML}, \quad J \equiv J_M = \{j : \mathbf{d}_j \neq 0\}, \quad \mathbf{d}_{J^c} = \mathbf{0}. \quad (3.2)$$

Consider a set of clustering matrices $\mathcal{C}(m, n, L)$ satisfying (2.1). At this point we do not impose any restrictions on the set of clustering matrices but later on we shall consider some special cases such as fixed membership (initial groups remain valid on the whole time interval) or limited change (only at most n_0 nodes can change their memberships between two consecutive time points).

We find m, J, \mathbf{d} and \mathbf{C} as a solution of the following penalized least squares optimization problem

$$(\hat{m}, \hat{J}, \hat{\mathbf{d}}, \hat{\mathbf{C}}) = \underset{m, J, \mathbf{d}, \mathbf{C}}{\text{argmin}} \left[\|\mathbf{a} - \mathbf{C} \mathbf{W}^T \mathbf{d}\|^2 + \text{Pen}(|J|, m) \right] \quad \text{s.t.} \quad \mathbf{d}_{J^c} = \mathbf{0}, \quad \mathbf{C} \in \mathcal{C}(m, n, L) \quad (3.3)$$

where \mathbf{a} is defined in (2.8), $\mathbf{d} \in \mathbb{R}^{ML}$, $\mathbf{W} \in \mathbb{R}^{ML \times ML}$, $M = m(m+1)/2$ and

$$\text{Pen}(|J|, m) = 11 \log(|\mathcal{C}(m, n, L)|) + \frac{11}{2} |J| \log \left(\frac{25 m^2 L}{|J|} \right). \quad (3.4)$$

Note that since minimization is carried out also with respect to m , optimization problem should be solved separately for every $m = 1, \dots, n$, yielding $\hat{\mathbf{d}}_M, \hat{\mathbf{C}}_M$ and \hat{J}_M . After that, one needs to select the value $\hat{M} = \hat{m}(\hat{m}+1)/2$ that delivers the minimum in (3.3), so that

$$\hat{\mathbf{d}} = \hat{\mathbf{d}}_{\hat{M}}, \quad \hat{\mathbf{C}} = \hat{\mathbf{C}}_{\hat{M}}, \quad \hat{J} = \hat{J}_{\hat{M}}. \quad (3.5)$$

Finally, due to (3.2), we set $\hat{\mathbf{W}} = (\mathbf{H} \otimes \mathbf{I}_{\hat{M}})$ and calculate

$$\hat{\mathbf{q}} = \hat{\mathbf{W}}^T \hat{\mathbf{d}}, \quad \hat{\boldsymbol{\theta}} = \hat{\mathbf{C}} \hat{\mathbf{q}}. \quad (3.6)$$

We obtain $\hat{\boldsymbol{\Lambda}}$ by packing vector $\hat{\boldsymbol{\theta}}$ into the tensor and taking the symmetries into account.

Denote the true value of tensor $\boldsymbol{\Lambda}$ by $\boldsymbol{\Lambda}^*$. Also, denote by m^* the true number of groups, by \mathbf{q}^* and $\boldsymbol{\theta}^*$ the true values of \mathbf{q} and $\boldsymbol{\theta}$ in (2.8) and by \mathbf{C}^* the true value of \mathbf{C} . Let $M^* = m^*(m^*+1)/2$ and $\mathbf{W}^* = (\mathbf{H} \otimes \mathbf{I}_{M^*})$ be true values of M and \mathbf{W} , respectively.

Note that vector $\boldsymbol{\theta}^*$ is obtained by vectorizing $\boldsymbol{\Lambda}^*$ and then removing the redundant entries. Then, it follows from (2.8) that

$$\mathbf{a} = \boldsymbol{\theta}^* + \boldsymbol{\xi} \quad \text{with} \quad \boldsymbol{\theta}^* = \mathbf{C}^* \mathbf{q}^* = \mathbf{C}^* \mathbf{W}^{*T} \mathbf{d}^*. \quad (3.7)$$

Due to the relation between the ℓ_2 and the Frobenius norms, one has

$$\|\boldsymbol{\theta} - \boldsymbol{\theta}^*\|^2 \leq \|\boldsymbol{\Lambda} - \boldsymbol{\Lambda}^*\|^2 \leq 2\|\boldsymbol{\theta} - \boldsymbol{\theta}^*\|^2, \quad (3.8)$$

and the following statement holds.

Theorem 1 *Consider a DSBM with a true matrix of probabilities $\boldsymbol{\Lambda}^*$ and estimator $\hat{\boldsymbol{\Lambda}}$ obtained according to (3.3)–(3.6). Then, for any $t > 0$, one has*

$$\mathbb{P} \left\{ \frac{\|\hat{\boldsymbol{\Lambda}} - \boldsymbol{\Lambda}^*\|^2}{n^2 L} \leq \min_{\substack{m, J, \mathbf{q} \\ \mathbf{C} \in \mathcal{C}(m, n, L)}} \left[\frac{6 \|\mathbf{C} \mathbf{W}^T \mathbf{d}_{(J)} - \boldsymbol{\theta}^*\|^2}{n^2 L} + \frac{4 \text{Pen}(|J|, m)}{n^2 L} \right] + \frac{38}{n^2 L} t \right\} \geq 1 - 9e^{-t} \quad (3.9)$$

and

$$\mathbb{E} \left(\frac{\|\hat{\boldsymbol{\Lambda}} - \boldsymbol{\Lambda}^*\|^2}{n^2 L} \right) \leq \min_{\substack{m, J, \mathbf{q} \\ \mathbf{C} \in \mathcal{C}(m, n, L)}} \left[\frac{6 \|\mathbf{C} \mathbf{W}^T \mathbf{d}_{(J)} - \boldsymbol{\theta}^*\|^2}{n^2 L} + \frac{4 \text{Pen}(|J|, m)}{n^2 L} + \frac{342}{n^2 L} \right], \quad (3.10)$$

where $\mathbf{d}_{(J)}$ is the modification of vector \mathbf{d} where all elements \mathbf{d}_j with $j \notin J$ are set to zero.

The proof of the Theorem is given in Section 7. Here, we just explain its idea. Note that if the values of m and C are fixed, the problem (3.3) reduces to a regression problem with a complexity penalty $\text{Pen}(|J|, m)$. Moreover, if J is known, the optimal estimator $\hat{\mathbf{d}}$ of \mathbf{d}^* is just a projection estimator. Indeed, denote $\boldsymbol{\Upsilon} = \mathbf{C} \mathbf{W}^T$ and let $\boldsymbol{\Upsilon}_J = (\mathbf{C} \mathbf{W}^T)_J$ be the reduction of matrix $\mathbf{C} \mathbf{W}^T$ to columns $j \in J$. Given \hat{m} , \hat{J} and $\hat{\mathbf{C}}$, one obtains $\hat{M} = \hat{m}(\hat{m}+1)/2$, $\hat{\mathbf{W}} = (\mathbf{H} \otimes \mathbf{I}_{\hat{M}})$, $\hat{\boldsymbol{\Upsilon}} = \hat{\mathbf{C}} \hat{\mathbf{W}}^T$ and $\hat{\boldsymbol{\Upsilon}}_J = (\hat{\mathbf{C}} \hat{\mathbf{W}}^T)_J$. Let

$$\Pi_J = \boldsymbol{\Upsilon}_J (\boldsymbol{\Upsilon}_J^T \boldsymbol{\Upsilon}_J)^{-1} \boldsymbol{\Upsilon}_J^T \quad \text{and} \quad \hat{\Pi}_{\hat{J}} = \hat{\boldsymbol{\Upsilon}}_{\hat{J}} (\hat{\boldsymbol{\Upsilon}}_{\hat{J}}^T \hat{\boldsymbol{\Upsilon}}_{\hat{J}})^{-1} \hat{\boldsymbol{\Upsilon}}_{\hat{J}}^T \quad (3.11)$$

be projection matrices on the column spaces of $\mathbf{\Upsilon}_J$ and $\hat{\mathbf{\Upsilon}}_{\hat{J}}$, respectively. Then, it is easy to see that $\hat{\mathbf{\Upsilon}}\hat{\mathbf{d}} = \hat{\Pi}_J\mathbf{a}$ and vector $\hat{\mathbf{d}}$ is of the form

$$\hat{\mathbf{d}} = (\hat{\mathbf{\Upsilon}}_{\hat{J}}^T \hat{\mathbf{\Upsilon}}_{\hat{J}})^{-1} \hat{\mathbf{\Upsilon}}_{\hat{J}}^T \mathbf{a}. \quad (3.12)$$

Hence, the values of \hat{m} , \hat{J} and $\hat{\mathbf{C}}$ can be obtained as a solution of the following optimization problem

$$(\hat{\mathbf{C}}, \hat{m}, \hat{J}) = \underset{m, J, \mathbf{C}}{\operatorname{argmin}} [\|\mathbf{a} - \Pi_J \mathbf{a}\|^2 + \operatorname{Pen}(|J|, m)] \quad \text{s.t. } \mathbf{C} \in \mathcal{C}(m, n, L),$$

where Π_J and the penalty $\operatorname{Pen}(|J|, m)$ are defined in (3.11) and (3.4), respectively. After that, the proof uses arguments that are relatively standard for the penalized least squares estimation.

Note that in the right-hand sides of expressions (3.9) and (3.10), $\|\mathbf{C}\mathbf{W}^T \mathbf{d}_{(J)} - \boldsymbol{\theta}^*\|^2$ is the bias term that quantifies how well one can estimate the true values of probabilities $\boldsymbol{\theta}^*$ by blocking them together, averaging the values in each block and simultaneously setting all but $|J|$ coefficients of the expansions of the vector \mathbf{q} in (2.8) in the basis \mathbf{W} to zero. The penalty constitutes the "price" for choosing too many blocks and coefficients. In particular, the second term in (3.4), $(n^2 L)^{-1} |J| \log(4m^2 L/|J|)$ is due to the need of finding and estimating $|J|$ elements of the $Lm(m+1)/2$ -dimensional vector. The first term, $\log(|\mathcal{C}(m, n, L)|)$, accounts for the difficulty of clustering and is due to application of the union bound in probability.

Theorem 1 holds for any collection $\mathcal{C}(m, n, L)$ of clustering matrices. Denote by $\mathcal{Z}(m, n, n_0, L)$ the collection of clustering matrices corresponding to the situation where at most n_0 nodes can change their memberships between any two consecutive time points, so that

$$|\mathcal{Z}(m, n, n_0, L)| = m^n \left[\binom{n}{n_0} m^{n_0} \right]^{L-1}; \quad |\mathcal{Z}(m, n, 0, L)| = m^n; \quad |\mathcal{Z}(m, n, n, L)| = m^{nL}. \quad (3.13)$$

Note that the case of $n_0 = 0$ corresponds to the scenario where the group memberships of the nodes are constant and do not depend on time while the case of $n_0 = n$ means that memberships of all nodes can change arbitrarily from one time instant to another.

Corollary 1 *Consider a DSBM with a true matrix of probabilities $\mathbf{\Lambda}^*$ and estimator $\hat{\mathbf{\Lambda}}$ obtained according to (3.3)–(3.6) where $\mathcal{C}(m, n, L) = \mathcal{Z}(m, n, n_0, L)$. Then, inequalities (3.9) and (3.10) hold with*

$$\frac{\operatorname{Pen}(|J|, m)}{n^2 L} = \frac{11 \log m}{nL} + \frac{11 n_0(L-1)}{n^2 L} \log \left(\frac{mne}{n_0} \right) + \frac{11 |J|}{2n^2 L} \log \left(\frac{25 m^2 L}{|J|} \right) \quad (3.14)$$

Remark 1 (Sparse SBM). Theorem 1 provides an oracle inequality in the case of a time-independent SBM ($L = 1$). Indeed, in this case, by taking $\mathbf{H} = \mathbf{1}$ and $\mathbf{W} = \mathbf{I}_M$, obtain for any $t > 0$

$$\frac{1}{n^2} \mathbb{E} \|\hat{\mathbf{\Lambda}} - \mathbf{\Lambda}^*\|^2 \leq \min_{\substack{m, J, \mathbf{q} \\ \mathbf{C} \in \mathcal{M}(m, n)}} \left[\frac{6 \|\mathbf{C}\mathbf{q}_{(J)} - \boldsymbol{\theta}^*\|^2}{n^2} + \frac{44 \log m}{n} + \frac{22|J|}{n^2} \log \left(\frac{25 m^2}{|J|} \right) \right] + \frac{342}{n^2} \quad (3.15)$$

and a similar result holds for the probability. Note that if $|J| = m(m+1)/2$, our result coincides with the one of Gao *et al.* (2015). However, when $|J|$ is small, the right-hand of (3.15) may be smaller than $C(\log m/n + m^2/n^2)$ obtained in Gao *et al.* (2015). In addition, the estimator obtained above is adaptive to the unknown number of classes. In particular, (3.15) automatically provides an upper bound for the error when the network is sparse and only $|J|$ pairs of classes have a nonzero probability of connection.

4 The lower bounds for the risk for the DSBM

In order to prove that the estimator obtained as a solution of optimization problem (3.3) is optimal, we need to show that the minimax lower bound for the mean squared error of any estimator of $\mathbf{\Lambda}$ is proportional to $\text{Pen}(|J|, m)$ defined in (3.4). In order to be more specific, we consider the collection of clustering matrices $\mathcal{Z}(m, n, n_0, L)$ with cardinality given by (3.13) that corresponds to the situation where at most n_0 nodes can change their memberships between consecutive time instants. In this case, $\text{Pen}(|J|, m)$ is defined in (3.14). In order to derive the lower bounds for the error, we impose mild conditions on the orthogonal matrix \mathbf{H} : for any binary vector $\boldsymbol{\omega} \in \{0, 1\}^L$ one has

$$\|\mathbf{H}^T \boldsymbol{\omega}\|_\infty \leq \|\boldsymbol{\omega}\|_1 / \sqrt{L} \quad \text{and} \quad \mathbf{H} \mathbf{1} = \sqrt{L} \mathbf{e}_1, \quad (4.1)$$

where $\mathbf{1} = (1, 1, \dots, 1)^T$ and $\mathbf{e}_1 = (1, 0, \dots, 0)^T$. Assumptions (4.1) are not restrictive. In fact, they are satisfied for a variety of common orthogonal transforms such as Fourier or wavelet transforms. Let $\mathcal{G}_{m,L,s}$ be a collection of tensors such that $\mathbf{G} \in \mathcal{G}_{m,L,s}$ implies that the vectorized versions \mathbf{q} of \mathbf{G} can be written as $\mathbf{q} = \mathbf{W}^T \mathbf{d}$ with $\|\mathbf{d}\|_0 \leq s$.

Theorem 2 *Let orthogonal matrix \mathbf{H} satisfy condition (4.1). Consider the DSBM where $\mathbf{G} \in \mathcal{G}_{m,L,s}$ with $s \geq \kappa m^2$ where $\kappa > 0$ is independent of m, n and L . Denote $\gamma = \min(\kappa, 1/2)$ and assume that $L \geq 2, n \geq 2m, n_0 \leq \min(\gamma n, 4/3 \gamma n m^{-1/9})$ and*

$$s^2 \log(2LM/s) \leq 68LMn^2. \quad (4.2)$$

Then

$$\inf_{\hat{\mathbf{\Lambda}}} \sup_{\substack{\mathbf{G} \in \mathcal{G}_{m,L,s} \\ \mathbf{C} \in \mathcal{Z}(m,n,n_0,L)}} \mathbb{P}_{\mathbf{\Lambda}} \left\{ \frac{\|\hat{\mathbf{\Lambda}} - \mathbf{\Lambda}\|^2}{n^2 L} \geq C_\gamma \left(\frac{\log m}{nL} + \frac{n_0}{n^2} \log \left(\frac{mne}{n_0} \right) + \frac{s \log(Lm^2/s)}{n^2 L} \right) \right\} \geq \frac{1}{4}, \quad (4.3)$$

where $\hat{\mathbf{\Lambda}}$ is any estimator of $\mathbf{\Lambda}$, C_γ is an absolute constant that depends on γ only.

Note that the terms $\log m/(nL)$ and $n_0 n^{-2} \log(mne/n_0)$ in (4.3) correspond to, respectively, the error of initial clustering and the clustering error due to membership changes. The third term $(n^2 L)^{-1} s \log(Lm^2/s)$ is due to nonparametric estimation and model selection. Condition that $s \geq \kappa m^2$ for some $\kappa > 0$ independent of m, n and L , ensures that one does not have too many classes where nodes have no interactions with each other or members of other classes while (4.2) is just a technical condition which ensures that elements of tensor $\mathbf{\Lambda}$ are probabilities.

If $\mathbf{G} \in \mathcal{G}_{m,L,s}$ and $\mathbf{C} \in \mathcal{Z}(m, n, n_0, L)$, then the lower bound for the error in (4.3) coincides, up to a constant, with the penalty in (3.14). Since the bias term is equal to zero for the true tensor of probabilities $\mathbf{\Lambda}^*$, the latter means that the estimator constructed above is minimax optimal up to a constant.

5 Dynamic graphon estimation

Consider the situation where tensor $\mathbf{\Lambda}$ is generated by a dynamic graphon f , so that $\mathbf{\Lambda}$ is given by expression (1.1) where function $f : [0, 1]^3 \rightarrow [0, 1]$ is such that $f(x, y, t) = f(y, x, t)$ for any t and $\boldsymbol{\zeta} = (\zeta_1, \dots, \zeta_n)$ is a random vector sampled from a distribution $\mathbb{P}_{\boldsymbol{\zeta}}$ supported on $[0, 1]^n$. We further assume that probabilities $\mathbf{\Lambda}_{i,j,l}$ do not change drastically from one time point to another, i.e. that f is smooth in t . We shall also assume that f is piecewise smooth in x and y .

In order to quantify those assumptions, for each $x, y \in [0, 1]^2$, we consider a vector $\mathbf{f}(x, y) = (f(x, y, t_1), \dots, f(x, y, t_L))^T$ and an orthogonal transform \mathbf{H} used in the previous sections. We assume that elements $\mathbf{v}_l(x, y)$ of vector $\mathbf{v}(x, y) = \mathbf{H}\mathbf{f}(x, y)$ satisfy the following assumption:

(A). There exist constants $0 = \beta_0 < \beta_1 < \dots < \beta_r = 1$ and $\nu_1, \nu_2, K_1, K_2 > 0$ such that for any $x, x' \in (\beta_{i-1}, \beta_i]$ and $y, y' \in (\beta_{j-1}, \beta_j]$, $1 \leq i, j \leq r$, one has

$$[\mathbf{v}_l(x, y) - \mathbf{v}_l(x', y')]^2 \leq K_1[|x - x'| + |y - y'|]^{2\nu_1}, \quad (5.1)$$

$$\sum_{l=2}^L l^{2\nu_2} \mathbf{v}_l^2(x, y) \leq K_2. \quad (5.2)$$

Note that, for a graphon corresponding to the DSBM model, on each of the rectangles $(\beta_{i-1}, \beta_i] \times (\beta_{j-1}, \beta_j]$, functions $\mathbf{v}_l(x, y)$ are constant, so that $\nu_1 = \infty$. The sum in (5.2) starts from $l = 2$ since, for majority of transforms, the term $\mathbf{v}_1(x, y)$ returns the average of the $f(x, y, t)$ over $t \in [0, 1]$ and does not need to be penalized. Moreover, if one simply considers values of l starting from zero, conditions $\sum_{l=1}^{L-1} l^{2\nu_2} \mathbf{v}_l^2(x, y) \leq K_2$ and $\sum_{l=0}^{L-1} l^{2\nu_2} \mathbf{v}_l^2(x, y) \leq K_2$ become equivalent.

We denote the class of graphons satisfying assumptions (1.1)–(5.2) by $\Sigma(\nu_1, \nu_2, K_1, K_2)$. Note that, since ν_1, ν_2, K_1 and K_2 in Assumption **A** are independent of x and y , one can simplify optimization procedure as follows. Denote $\mathbf{V} = \mathbf{Q}\mathbf{H}^T$. Since the memberships do not change with time, matrices $\mathbf{C}^{(l)}$ are independent of l , so that $\mathbf{C}^{(l)} = \mathbf{Z}$. Then, $\mathbf{\Theta} = \mathbf{Z}\mathbf{Q} = \mathbf{Z}\mathbf{V}\mathbf{H}$. Denote $\mathbf{\Phi} = \mathbf{\Theta}\mathbf{H}^T = \mathbf{Z}\mathbf{V}$. Apply transform \mathbf{H} to the data matrix \mathbf{A} obtaining matrix $\mathbf{X} = \mathbf{A}\mathbf{H}^T$. Choose $\rho \in (0, 1]$ and remove all columns $\mathbf{X}_{*,l}$ with $l > L^\rho$ obtaining matrix $\mathbf{X}^{(\rho)}$ with $\mathbb{E}\mathbf{X}^{(\rho)} = \mathbf{Z}\mathbf{V}^{(\rho)} \equiv \mathbf{\Phi}^{(\rho)}$ where matrix $\mathbf{V}^{(\rho)}$ has L^ρ columns. Since $|J| = ML^\rho \leq m^2 L^\rho$, optimization procedure (3.3) in this case can be reformulated as

$$(\hat{m}, \hat{\rho}, \hat{\mathbf{V}}^{(\hat{\rho})}, \hat{\mathbf{Z}}) = \underset{\substack{m, \rho, \mathbf{V}^{(\rho)} \\ \mathbf{Z} \in \mathcal{Z}(m, n, 0, L)}}{\operatorname{argmin}} \left[\|\mathbf{X}^{(\rho)} - \mathbf{Z}\mathbf{V}^{(\rho)}\|^2 + 11n \log m + \frac{11}{2} m^2 L^\rho \log(25 L^{1-\rho}) \right] \quad (5.3)$$

where $\mathcal{Z}(m, n, 0, L)$ is defined in (3.13). After that, set $\hat{\mathbf{\Theta}} = \mathbf{Z}\mathbf{V}^{(\hat{\rho})}\mathbf{H}$ and obtain $\hat{\mathbf{\Lambda}}$ by packing $\hat{\mathbf{\Theta}}$ into a tensor. The following corollary provides a minimax upper bound for the risk of the estimator $\hat{\mathbf{\Lambda}}$.

Theorem 3 *Let $\hat{\mathbf{\Lambda}}$ be obtained as a solution of optimization problem (5.3) as described above. Then, for $\Sigma \equiv \Sigma(\nu_1, \nu_2, K_1, K_2)$, one has*

$$\sup_{f \in \Sigma} \frac{\mathbb{E}\|\hat{\mathbf{\Lambda}} - \mathbf{\Lambda}^*\|^2}{n^2 L} \leq C \min_{\substack{1 \leq h \leq n-r \\ 0 \leq \rho \leq 1}} \left\{ \frac{L^{\rho-1}}{h^{2\nu_1}} + \frac{I(\rho < 1)}{L^{2\rho\nu_2+1}} + \frac{(h+r)^2(1+\log L)}{n^2 L^{1-\rho}} + \frac{\log(h+r)}{nL} \right\}, \quad (5.4)$$

where constant C depends on ν_1, ν_2, K_1 and K_2 only.

Note that construction of the estimator $\hat{\mathbf{\Lambda}}$ does not require knowledge of ν_1, ν_2, K_1 and K_2 , so the estimator is fully adaptive. On the other hand, an expression in the right hand side of (5.4) is rather complex and hard to analyze. For this reason, we shall consider only two regimes: $r = r_{n,L} \geq 2$ may depend on n or L and $\nu_1 = \infty$; or $r = r_0 \geq 1$ is a fixed quantity independent of n and L . The first regime corresponds to a piecewise constant (in x and y) graphon that generates the DSBM while the second regime deals with the situation where f is a piecewise smooth function of all three arguments. In the first case, we set $h = 2$ and, in the second, choose h to be a function of n and L . By minimizing the right-hand side of (5.4), we obtain the following statement.

Corollary 2 Let $\widehat{\mathbf{\Lambda}}$ be obtained as a solution of optimization problem (5.3) as described above. Then, for $\Sigma \equiv \Sigma(\nu_1, \nu_2, K_1, K_2)$ and C independent of n and L , one has

$$\sup_{f \in \Sigma} \frac{\mathbb{E} \|\widehat{\mathbf{\Lambda}} - \mathbf{\Lambda}^*\|^2}{n^2 L} \leq \begin{cases} C \min \left\{ \frac{1}{L} \left[\left(\frac{r}{n} \right)^2 \log \left(\frac{n}{r} \right) \right]^{\frac{2\nu_2}{2\nu_2+1}} ; \left(\frac{r}{n} \right)^2 \right\} + \frac{C \log r}{nL}, & \text{if } r = r_{n,L}; \\ C \min \left\{ \frac{1}{L} \left(\frac{\log L}{n^2} \right)^{\frac{2\nu_1 \nu_2}{(\nu_1+1)(2\nu_2+1)}} ; \left(\frac{\log L}{n^2} \right)^{\frac{\nu_1}{\nu_1+1}} \right\} + \frac{C \log n}{nL}, & \text{if } r = r_0. \end{cases} \quad (5.5)$$

In order to assess optimality of the penalized least squares estimator obtained above, we derive lower bounds for the minimax risk over the set $\Sigma(\nu_1, \nu_2, K_1, K_2)$. These lower bounds are constructed separately for each of the two regimes.

Theorem 4 Let matrix \mathbf{H} satisfy assumptions (4.1) and $\nu_2 \geq 1/2$ in (5.2). Then, for C independent of n and L , one has

$$\inf_{\widehat{\mathbf{\Lambda}}} \sup_{f \in \Sigma(\nu_1, \nu_2, K_1, K_2)} \mathbb{P}_{\mathbf{\Lambda}} \left\{ \frac{\|\widehat{\mathbf{\Lambda}} - \mathbf{\Lambda}\|^2}{n^2 L} \geq \Delta(n, L) \right\} \geq \frac{1}{4}, \quad (5.6)$$

where

$$\Delta(n, L) = \begin{cases} C \min \left\{ \frac{1}{L} \left[\left(\frac{r}{n} \right)^2 \right]^{\frac{2\nu_2}{2\nu_2+1}} ; \left(\frac{r}{n} \right)^2 \right\} + \frac{C \log r}{nL}, & \text{if } r = r_{n,L}; \\ C \min \left\{ \frac{1}{L} \left(\frac{1}{n^2} \right)^{\frac{2\nu_1 \nu_2}{(\nu_1+1)(2\nu_2+1)}} ; \left(\frac{1}{n^2} \right)^{\frac{\nu_1}{\nu_1+1}} \right\} + \frac{C \log n}{nL}, & \text{if } r = r_0. \end{cases} \quad (5.7)$$

It is easy to see that the value of $\Delta(n, L)$ coincides with the upper bound in (5.5) up to a at most a logarithmic factor of n/r or L . In both cases, the first quantities in the minimums correspond to the situation where f is smooth enough as a function of time, so that application of transform \mathbf{H} improves estimation precision by reducing the number of parameters that needs to be estimated. The second quantities represent the case where one needs to keep all elements of vector \mathbf{d} and hence application of the transform yields no benefits. The latter can be due to the fact that ν_2 is too small or L is too low.

The upper and the lower bounds in Theorems 3 and 4 look somewhat similar to the ones appearing in anisotropic functions estimation (see, e.g., Lepski (2015)). Note also that although in the case of a stationary graphon ($L = 1$), the estimation precision does not improve if $\nu_1 > 1$, this is not the case in the case of a dynamic graphon. Indeed, the right-hand sides in (5.7) can be significantly smaller when ν_1, ν_2 and L are large.

Remark 2 (DSBM and dynamic graphon). Observe that since we assume that vector $\boldsymbol{\zeta}$ is independent of t , our set up corresponds to the situation where the nodes of the network do not change their memberships in time and $n_0 = 0$ in (3.13). Therefore, a piecewise constant graphon is just a particular case of a general DSBM since the latter allows any temporal changes of nodes' memberships. On the other hand, the dynamic piecewise constant graphon formulation allows to derive specific minimax convergence rates for the model.

6 Extensions and generalizations

In the present paper we considered estimation of connection probabilities in the context of dynamic network models. Our approach is based on vectorization of the model and allows a variety of extensions.

1. **(Inhomogeneous or non-smooth connection probabilities).** Although assumption (5.2) essentially implies that probabilities of connections are represented by smooth functions of time that belong to the same Sobolev class, one can easily extend our theory to the case where those probabilities have jump discontinuities and belong to different functional classes. Indeed, by generalizing condition (5.2) and assuming that \mathbf{H} is a wavelet transform, one can accommodate this case similarly to how this was done in Klopp and Pensky (2015).
2. **(Time-dependent number of nodes).** One can apply the theory above even when the number of nodes in the network changes from one time instant to another. Indeed, in this case we can form a set that includes all nodes which have ever been in the network and denote their number by n . Consider a class Ω_0 such that all nodes in this class have zero probability of interaction with each other or any other node in the network. At each time instant, place all nodes that are not in the network into the class Ω_0 . After that, one just needs to modify the optimization procedures by placing additional restrictions that out-of-the-network nodes indeed belong to class Ω_0 and that $\mathbf{G}_{0,k,l} = 0$ for any $k = 0, 1, 2, \dots, m$ and $l = 1, \dots, L$.
3. **(Adaptivity to clustering complexity).** Although, in the case of the DSBM model, our estimator is adaptive to the unknown number of classes, it requires knowledge about the extent to which nodes' memberships can change from one time instant to another. For example, if at most n_0 nodes can change their memberships between two consecutive time points and n_0 is a fixed quantity independent of n and m , we can replace n_0 by $\log n$ that dominates n_0 if n is large enough. However, if n_0 depends on n and m , development of an adaptive estimator requires additional investigation.
4. **(More general dynamic graphon function).** In the present paper, we consider graphon function which is based on a time-independent random vector $\boldsymbol{\zeta} = (\zeta_1, \dots, \zeta_n)$ sampled from a distribution $\mathbb{P}_{\boldsymbol{\zeta}}$ supported on $[0, 1]^n$ and connection probabilities are defined by (1.1). This model corresponds to the case where nodes do not change their memberships in time. It is possible, however, to consider a more diverse scenario where there exists a latent random field $\boldsymbol{\zeta}(t) = (\zeta_1(t), \dots, \zeta_n(t))$ and probability of connection is given by $\mathbf{A}_{i,j,l} = f(\zeta_i(t_l), \zeta_j(t_l), t_l)$, $i, j = 1, \dots, n$, $l = 1, \dots, L$. The latter model is much more complex since one needs to answer a question of how assumptions about trajectories of $\zeta_i(t)$ affect the behavior of f .

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7 Proofs

7.1 Proofs of the upper bounds for the risk

Proof of Theorem 1. Since $(\hat{\mathbf{d}}, \hat{\mathbf{C}}, \hat{m}, \hat{J})$ are solutions of optimization problem (3.3), for any m , J , \mathbf{C} and \mathbf{d} one has

$$\|\mathbf{a} - \hat{\mathbf{C}}\hat{\mathbf{W}}^T \hat{\mathbf{d}}_{(\hat{J})}\|^2 + \text{Pen}(|\hat{J}|, \hat{m}) \leq \|\mathbf{a} - \mathbf{C}\mathbf{W}^T \mathbf{d}_{(J)}\|^2 + \text{Pen}(|J|, m), \quad (7.1)$$

For any m , J , \mathbf{C} and \mathbf{d} , it follows from (3.7) that

$$\begin{aligned} \|\mathbf{a} - \mathbf{C}\mathbf{W}^T \mathbf{d}_{(J)}\|^2 &= \|(\mathbf{a} - \mathbf{C}^* \mathbf{W}^{*T} \mathbf{d}^*) + (\mathbf{C}^* \mathbf{W}^{*T} \mathbf{d}^* - \mathbf{C}\mathbf{W}^T \mathbf{d}_{(J)})\|^2 \\ &= \|\boldsymbol{\xi}\|^2 + \|\mathbf{C}\mathbf{W}^T \mathbf{d}_{(J)} - \mathbf{C}^* \mathbf{W}^{*T} \mathbf{d}^*\|^2 + 2\boldsymbol{\xi}^T (\mathbf{C}^* \mathbf{W}^{*T} \mathbf{d}^* - \mathbf{C}\mathbf{W}^T \mathbf{d}_{(J)}). \end{aligned}$$

Hence, plugging the last identity into the inequality (7.1), derive that for any m , J , \mathbf{C} and \mathbf{d}

$$\|\hat{\mathbf{C}}\hat{\mathbf{W}}^T \hat{\mathbf{d}}_{(\hat{J})} - \mathbf{C}^* \mathbf{W}^{*T} \mathbf{d}^*\|^2 \leq \|\mathbf{C}\mathbf{W}^T \mathbf{d}_{(J)} - \mathbf{C}^* \mathbf{W}^{*T} \mathbf{d}^*\|^2 + 2\Delta + \text{Pen}(|J|, m) - \text{Pen}(|\hat{J}|, \hat{m}). \quad (7.2)$$

Here

$$\Delta = 2|\boldsymbol{\xi}^T (\hat{\mathbf{C}}\hat{\mathbf{W}}^T \hat{\mathbf{d}}_{(\hat{J})} - \mathbf{C}\mathbf{W}^T \mathbf{d}_{(J)})| \leq \Delta_1 + \Delta_2 + \Delta_3, \quad (7.3)$$

$$\Delta_1 = 2|\boldsymbol{\xi}^T (\mathbf{C}^* \mathbf{W}^{*T} \mathbf{d}^* - \mathbf{C}\mathbf{W}^T \mathbf{d}_{(J)})|, \quad \Delta_2 = 2|\boldsymbol{\xi}^T (\mathbf{I}_{NL} - \hat{\Pi}_{\hat{J}}) \mathbf{C}^* \mathbf{W}^{*T} \mathbf{d}^*|, \quad \Delta_3 = 2\boldsymbol{\xi}^T \hat{\Pi}_{\hat{J}} \boldsymbol{\xi}, \quad (7.4)$$

since, due to (3.11) and (3.12), one has $\hat{\mathbf{C}}\hat{\mathbf{W}}^T \hat{\mathbf{d}}_{(\hat{J})} = \hat{\Pi}_{\hat{J}} \mathbf{a}$ where \mathbf{a} is given by (3.7). Now, we need to find upper bounds for each of the terms in (7.4).

By Lemma 1 with $\alpha = 1/2$ and any $t > 0$, one has

$$\mathbb{P} \left\{ \Delta_1 - 0.5 \|\boldsymbol{\xi}^T (\mathbf{C}^* \mathbf{W}^{*T} \mathbf{d}^* - \mathbf{C}\mathbf{W}^T \mathbf{d}_{(J)})\|^2 \leq 4t \right\} \geq 1 - 2e^{-t}. \quad (7.5)$$

Note that

$$\|\hat{\mathbf{C}}\hat{\mathbf{W}}^T \hat{\mathbf{d}}_{(\hat{J})} - \mathbf{C}^* \mathbf{W}^{*T} \mathbf{d}^*\|^2 = \|\hat{\Pi}_{\hat{J}} \boldsymbol{\xi}\|^2 + \|(\mathbf{I}_{NL} - \hat{\Pi}_{\hat{J}}) \mathbf{C}^* \mathbf{W}^{*T} \mathbf{d}^*\|^2 \geq \|(\mathbf{I}_{NL} - \hat{\Pi}_{\hat{J}}) \mathbf{C}^* \mathbf{W}^{*T} \mathbf{d}^*\|^2.$$

Therefore, applying Lemma 1 with $\alpha = 1/2$ and an union bound over $m = 1, \dots, n$, $\mathbf{C} \in \mathcal{C}(m, n, L)$ and J with $|J| = 1, \dots, ML$ derive that for any $t > 0$

$$\mathbb{P} \left\{ \Delta_2 - 0.5 \|\hat{\mathbf{C}}\hat{\mathbf{W}}^T \hat{\mathbf{d}}_{(\hat{J})} - \mathbf{C}^* \mathbf{W}^{*T} \mathbf{d}^*\|^2 - 4R(\hat{m}, \hat{J}, L) \leq 4t \right\} \geq 1 - 6e^{-t}, \quad (7.6)$$

where

$$R(m, J, L) = \log(|\mathcal{C}(m, n, L)|) + |J| \log(\hat{m}^2 L e / |J|) + 2 \log(m|J|). \quad (7.7)$$

Finally, in order to obtain an upper bound for Δ_3 , apply Lemma 2 with $\mathbf{A} = \hat{\Pi}_{\hat{J}}$ and again use the union upper bound over $m = 1, \dots, n$, $\mathbf{C} \in \mathcal{C}(m, n, L)$ and J with $|J| = 1, \dots, ML$. Since for any projection matrix Π_J , one has $\|\Pi_J\| = 1$ and $\|\Pi_J\|_F^2 = |J|$, derive for any $t > 0$ that

$$\mathbb{P} \left\{ \Delta_3 - |\hat{J}| - \frac{3}{2} R(\hat{m}, \hat{J}, L) \leq \frac{3t}{2} \right\} \geq 1 - e^{-t}, \quad (7.8)$$

where $R(m, J, L)$ is defined in (7.7). Combining (7.2)–(7.8) and recalling that $\hat{\boldsymbol{\theta}} = \hat{\mathbf{C}}\hat{\mathbf{W}}^T \hat{\mathbf{d}}_{(\hat{J})}$ and $\boldsymbol{\theta}^* = \mathbf{C}^* \mathbf{W}^{*T} \mathbf{d}^*$, obtain

$$\mathbb{P} \left\{ \|\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}^*\|^2 \leq \min_{\substack{m, J, \mathbf{q} \\ \mathbf{C} \in \mathcal{C}(m, n, L)}} [3 \|\mathbf{C}\mathbf{W}^T \mathbf{d}_{(J)} - \boldsymbol{\theta}^*\|^2 + 2 \text{Pen}(|J|, m)] + 19t \right\} \geq 1 - 9e^{-t}$$

provided

$$2\text{Pen}(|J|, m) \geq 11R(m, J, L) + 2|J|. \quad (7.9)$$

In order to complete the proof of (3.9), apply formula (3.8) and note that $\log(|\mathcal{C}(m, n, L)|) \geq \log(|\mathcal{M}(m, n)|) = n \log m > 2 \log m$ for $n \geq 2$ and $2 \log(|J|) \leq 2|J|$, so that $\text{Pen}(|J|, m)$ in (3.4) guarantees (7.9).

Finally, inequality (3.10) can be proved by noting that for any random variable ζ one has $\mathbb{E}\zeta \leq \int_0^\infty \mathbb{P}(\zeta > z) dz$.

Proof of Theorem 3. To prove (5.4), we use inequality (3.10) in Theorem 1. In order to find an upper bound for the bias term, we need to cluster the nodes and create an approximate connectivity tensor \mathbf{Q} . For this purpose, let h be a positive integer and denote $\kappa_j = 1 + [(\beta_j - \beta_{j-1})h]$ where $[x]$ is the largest integer no larger than x . For $k = 1, \dots, \kappa_j$ and $j = 1, \dots, r$, consider a set of intervals

$$U_{j,k} = (U_{j,k}^{(1)}, U_{j,k}^{(2)}] \quad \text{with} \quad U_{j,k}^{(1)} = \beta_{j-1} + (k-1)/h, \quad U_{j,k}^{(2)} = \min\{\beta_{j-1} + k/h, \beta_j\}.$$

Intervals $U_{j,k}$ subdivide every interval $(\beta_{j-1}, \beta_j]$ into κ_j sub-intervals of length at most $1/h$ and the total number of intervals is equal to

$$m = \sum_{j=1}^r \kappa_j.$$

Re-number the intervals consecutively as U_1, \dots, U_m and observe that since $(\beta_j - \beta_{j-1})h \leq \kappa_j \leq 1 + (\beta_j - \beta_{j-1})h$, one has

$$h \leq m \leq h + r. \quad (7.10)$$

The value m in (7.10) acts as a number of classes. Indeed, if $\zeta_i \in U_k$, we place node i into class Ω_k and set $\mathbf{Z}_{i,j} = \mathbb{I}(j = k)$. Let Θ^* be the true tensor of connection probabilities. Set $\mathbf{Q} = (\mathbf{Z}^T \mathbf{Z})^{-1} \mathbf{Z}^T \Theta^*$ and $\mathbf{V} = \mathbf{Q} \mathbf{H}^T$.

Since \mathbf{H} is an orthogonal matrix, the bias term in oracle inequality (3.10) is equal to

$$\|\mathbf{Z} \mathbf{V}^{(\rho)} - \Theta \mathbf{H}^T\|_F^2 = \|\mathbf{Z} \mathbf{V}^{(\rho)} - \Phi\|_F^2 = \|\mathbf{Z} \mathbf{V}^{(\rho)} - \Phi^{(\rho)}\|_F^2 + \|\Phi - \Phi^{(\rho)}\|_F^2. \quad (7.11)$$

The upper bound for the first term can be found by repeating calculation of Gao *et al.* (2015) with the only difference that f is replaced by v_l and there is an additional sum over $l = 1, \dots, L^\rho$. Then, we obtain

$$\|\mathbf{Z} \mathbf{V}^{(\rho)} - \Phi^{(\rho)}\|_F^2 \leq K_1 2^{2\nu_1} n^2 L^\rho h^{-2\nu_1}. \quad (7.12)$$

On the other hand, if $\rho < 1$, then by Assumption **A**,

$$\|\Phi - \Phi^{(\rho)}\|_F^2 \leq \sum_{i_1, i_2=1}^n \sum_{l=L^\rho+1}^L \mathbf{v}_l^2(\zeta_{i_1}, \zeta_{i_2}) \leq K_2 n^2 L^{-2\nu_2 \rho} \quad (7.13)$$

and $\|\Phi - \Phi^{(\rho)}\|_F^2 = 0$ if $\rho = 1$. Now, note that for given m and ρ , one has $|J| \leq m^2 L^\rho$, so that

$$\text{Pen}(|J|, m) \leq C [n \log m + m^2 L^\rho \log(25 L^{1-\rho})]. \quad (7.14)$$

Therefore, (7.10)–(7.14) yield (5.4).

Proof of Corollary 2. Note that if $\nu_1 = \infty$, one can set $h = 2$, so that the first term in (5.4) is equal to zero, $h + r \leq 3r$ and

$$\frac{\mathbb{E}\|\hat{\mathbf{\Lambda}} - \mathbf{\Lambda}^*\|^2}{n^2 L} \leq C \left\{ \frac{I(\rho < 1)}{L^{2\rho\nu_2+1}} + \left(\frac{r}{n}\right)^2 \frac{\log L}{L^{1-\rho}} + \frac{\log r}{nL} \right\}.$$

Minimizing this expression with respect to $\rho \in [0, 1]$ obtain the result in (5.5) for $r = r_{n,L}$. If $r = r_0$, then

$$\frac{\mathbb{E}\|\hat{\mathbf{\Lambda}} - \mathbf{\Lambda}^*\|^2}{n^2 L} \leq C \left\{ \frac{L^{\rho-1}}{h^{2\nu_1}} + \frac{I(\rho < 1)}{L^{2\rho\nu_2+1}} + \frac{h^2 \log L}{n^2 L^{1-\rho}} + \frac{\log h}{nL} \right\}. \quad (7.15)$$

Minimizing (7.15) with respect to h and ρ , obtain that the values h^* and $L^* = L^{\rho^*}$ delivering the minimum in (7.15) are such that $h^* \asymp (n^2 / \log L)^{1/(2(\nu_1+1))}$. Hence, for some absolute constants C , one has $\log(h^*) \leq C \log n$ and

$$L^* = \min \left\{ L, C (n^2 / \log L)^{\frac{\nu_1}{(\nu_1+1)(2\nu_2+1)}} \right\}.$$

Therefore, (5.5) holds for $r = r_0$.

7.2 Proofs of the lower bounds for the risk

Proof of Theorem 2. The error consists of two parts, the clustering error and the nonparametric estimation error. We shall consider those terms separately.

The clustering error. Without loss of generality, assume that γm and γn are integers. Assume that connectivity tensor \mathbf{G} does not change with l , so $\mathbf{G}_{*,*,l} = \mathbf{V}$ is an $m \times m$ symmetric matrix. Let \mathbf{V} be block diagonal and such that the diagonal blocks are equal to zero and the non-diagonal blocks are equal to \mathbf{F} and \mathbf{F}^T , respectively, so that $\mathbf{G}_{i,j} = 0$ if $1 \leq i, j \leq (1-\gamma)m$ or $(1-\gamma)m+1 \leq i, j \leq m$ and $\mathbf{G}_{i,(1-\gamma)m+j} = \mathbf{F}_{i,j}$ if $i = 1, \dots, (1-\gamma)m$, $j = (1-\gamma)m+1, \dots, m$. Since components of vectors $\mathbf{G}_{k_1,k_2,*}$ are constant for any k_1, k_2 , due to condition (4.1), the set J has at most $\gamma(1-\gamma)m^2 < s$ nonzero elements.

Consider a collection of binary vectors $\boldsymbol{\omega} \in \{0, 1\}^{(1-\gamma)m}$. By Varshamov-Gilbert Lemma (see Tsybakov (2009)), there exists a subset Ξ of those vectors such that for any $\boldsymbol{\omega}, \boldsymbol{\omega}' \in \Xi$ one has $\|\boldsymbol{\omega} - \boldsymbol{\omega}'\|_H = \|\boldsymbol{\omega} - \boldsymbol{\omega}'\|^2 \geq (1-\gamma)m/8 \geq m/16$ and $|\Xi| \geq \exp((1-\gamma)m/8)$. Assume, without loss of generality, that m is large enough, so that $\exp((1-\gamma)m/8) \geq \gamma m$, otherwise, choose a smaller value of γ (inequality $\exp((1-\gamma)m/8) \geq \gamma m$ is always valid for $\gamma \leq 1/9$). Choose γm vectors $\boldsymbol{\omega}$ in Ξ , enumerate them as $\boldsymbol{\omega}^{(1)}, \dots, \boldsymbol{\omega}^{(\gamma m)}$ and use them as columns of matrix \mathbf{F} :

$$\mathbf{F}_{*,j} = 0.5 \mathbf{1} + \rho \boldsymbol{\omega}^{(j)}, \quad j = 1, \dots, \gamma m. \quad (7.16)$$

Then, for any $j, j' = 1, \dots, \gamma m$, obtain

$$\|\mathbf{F}_{*,j} - \mathbf{F}_{*,j'}\|^2 \geq \rho^2 m / 16. \quad (7.17)$$

Note that for every l and k one has

$$l \log \left(\frac{k}{l} \right) \leq \log \left(\frac{k}{l} \right) \leq l \log \left(\frac{ke}{l} \right), \quad \log(k!) = k \log k - k + \frac{1}{2} \log(2\pi k) + o(1), \quad (7.18)$$

where the $o(1)$ term is smaller than 1. Therefore,

$$n \log m + n_0(L-1) \log \left(\frac{mn}{n_0} \right) \leq \log |\mathcal{Z}(m, n, n_0, L)| \leq n \log m + n_0(L-1) \log \left(\frac{mne}{n_0} \right) \quad (7.19)$$

The term $n \log m$ in (7.19) is due to the initial clustering while the term $n_0(L-1) \log(mn/n_0)$ is due to temporal changes in the clusters' memberships.

In what follows, we shall utilize clustering functions $z^{(l)} : [n] \rightarrow [m]$ corresponding to clustering matrices $\mathbf{C}^{(l)}$ such that $z^{(l)}(j) = k$ iff at the moment t_l node j belongs to class Ω_k , $k = 1, \dots, m$.

Clustering error due to initial clustering. First consider the case when initial clustering error dominates. If $m = 2$ or takes a small value, the proof is almost identical to Gao *et al.* (2015). Hence, we shall skip this part and consider the case when m is large enough, so that $\gamma m \geq 2$.

Following Gao *et al.* (2015), we consider clustering matrices and clustering functions independent of l , so that $z^{(l)} \equiv z$. Consider a sub-collection of clustering matrices $\mathcal{F}(m, n, \gamma) \subset \mathcal{M}(m, n)$ such that they cluster the first $n(1-\gamma)$ nodes into the first $m(1-\gamma)$ classes uniformly and sequentially, n/m nodes in each class. The remaining γn nodes are clustered into the remaining γm classes, n/m nodes into each class. Then, by Lemma 4, $\log |\mathcal{F}(m, n, \gamma)| \geq \gamma n \log(\gamma m)/4$. Now, apply Lemma 3 with $r = \gamma n/32$. Derive that there exists a subset $\mathcal{S}(m, n, \gamma)$ of the set $\mathcal{F}(m, n, \gamma)$ such that, for any $\mathbf{C}, \mathbf{C}' \in \mathcal{S}(m, n, \gamma)$, one has $2 \{\#j : z(j) \neq z'(j)\} = \|\mathbf{C} - \mathbf{C}'\|_H \geq \gamma n/32$. Also, by (7.52),

$$\log |\mathcal{S}(m, n, \gamma)| \geq \frac{\gamma n}{4} \log(\gamma m) - \frac{\gamma n \log(32m\gamma e)}{32} = \frac{\gamma n}{16} \log(\gamma m). \quad (7.20)$$

Let $\mathbf{\Lambda}$ and $\mathbf{\Lambda}'$ be the tensors of probabilities corresponding to, respectively, clustering matrices $\mathbf{C}, \mathbf{C}' \in \mathcal{S}(m, n, \gamma)$ with related clustering functions z and z' . Then, by (7.17), due to the fact that the first $n(1-\gamma)$ nodes are clustered uniformly and sequentially, obtain

$$\begin{aligned} \|\mathbf{\Lambda} - \mathbf{\Lambda}'\|^2 &= 2L \sum_{i=1}^{n(1-\gamma)} \sum_{j=n(1-\gamma)+1}^n (\mathbf{F}_{z(i), z(j)} - \mathbf{F}_{z'(i), z'(j)})^2 = \frac{2Ln}{m} \sum_{k=1}^{m(1-\gamma)} \sum_{j=n(1-\gamma)+1}^n (\mathbf{F}_{k, z(j)} - \mathbf{F}_{k, z'(j)})^2 \\ &= \frac{2Ln}{m} \sum_{j=n(1-\gamma)+1}^n \|\mathbf{F}_{*, z(j)} - \mathbf{F}_{*, z'(j)}\|^2 \geq \frac{Ln\rho^2}{16} \{\#j : z(j) \neq z'(j)\}, \end{aligned}$$

so that

$$\|\mathbf{\Lambda} - \mathbf{\Lambda}'\|^2 \geq 2^{-9} Ln^2 \rho^2 \gamma. \quad (7.21)$$

On the other hand, if $\rho \leq 1/4$, then, by Proposition 4.2 of Gao *et al.* (2015), obtain that the Kullback divergence is bounded above

$$K(\mathbb{P}_{\mathbf{\Lambda}}, \mathbb{P}_{\mathbf{\Lambda}'}) \leq 8 \|\mathbf{\Lambda} - \mathbf{\Lambda}'\|^2 \leq 16\rho^2 n^2 L \gamma. \quad (7.22)$$

Set $\rho^2 = C_\rho \log(\gamma m)/nL$ and apply Theorem 2.5 of Tsybakov (2009). Observe that, due to (7.20) and (7.22), conditions of Theorem 2.5 are satisfied if C_ρ is a small enough absolute constant. Since $Ln^2 \rho^2 \gamma = C_\rho n \gamma \log(\gamma m)$ and $\log(\gamma m) \geq C(\gamma) \log m$ for some constant $C(\gamma)$ dependent on γ only, derive

$$\inf_{\widehat{\mathbf{\Lambda}}} \sup_{\substack{\mathbf{G} \in \mathcal{G}_{m, L, s} \\ \mathbf{C} \in \mathcal{S}(m, n, \gamma)}} \mathbb{P}_{\mathbf{\Lambda}} \left\{ \frac{\|\widehat{\mathbf{\Lambda}} - \mathbf{\Lambda}^*\|^2}{n^2 L} \geq \frac{C_\gamma \log m}{nL} \right\} \geq 1/4 \quad (7.23)$$

Clustering error due to changes in the memberships. Now, we consider the case when the clustering error which is due to the temporal changes in memberships dominates the error of initial clustering. Use the same construction for \mathbf{G} and \mathbf{F} as before. Consider the following collection of clustering matrices $\mathcal{F} = \prod_{l=1}^L \mathcal{F}_l$ where \mathcal{F}_l is defined as follows. When l is odd, \mathcal{F}_l contains only one matrix that clusters nodes uniformly and sequentially, i.e., the first n/m nodes go to class Ω_1 , the second n/m nodes go to class Ω_2 and the last n/m nodes go to class Ω_m . If l is even, $\mathcal{F}_l = \mathcal{P}(m, n, n_0, \gamma)$ where $\mathcal{P}(m, n, n_0, \gamma)$ is the set of clustering matrices that corresponds to a perturbation of the uniform consecutive clustering with at most n_0 nodes moved to different classes in the manner described below. Let k_0 be an integer such that

$$k_0 \leq n_0/(\gamma m) < k_0 + 1. \quad (7.24)$$

If $k_0 = 0$, then $n_0 < \gamma m$ and we choose n_0 clusters out of the last γm clusters, choose one element in each of those clusters and then put those n_0 elements back in such a manner that every element goes to a different cluster and no elements goes back to its own cluster. If $k_0 \geq 1$, we choose k_0 elements in each of the last γm clusters and then put each of those k_0 -tuples back, one tuple per cluster, so that none of the tuple goes back to its own cluster. Then, $\log |\mathcal{F}| = [L/2] \log |\mathcal{P}(m, n, n_0, \gamma)|$ where $[L/2] \geq (L-1)/2$ is the largest integer not exceeding $L/2$ and

$$\log |\mathcal{P}(m, n, n_0, \gamma)| = \begin{cases} \log \binom{\gamma m}{n_0} + n_0 \log(n/m) + \log[(n_0 - 1)!], & \text{if } k_0 = 0; \\ \gamma m \log \binom{n/m}{k_0} + \log[(\gamma m - 1)!], & \text{if } k_0 \geq 1 \end{cases}$$

If $k_0 = 0$, so that $n_0 < \gamma m$, then, by (7.18), obtain that $\log |\mathcal{P}(m, n, n_0, \gamma)| \geq n_0 \log(\gamma m/n_0) + n_0 \log(n/m) = n_0 \log(\gamma n/n_0)$. If $n_0 \geq \gamma m$ and $k_0 \geq 1$, then, by (7.18), obtain $\log |\mathcal{P}(m, n, n_0, \gamma)| \geq \gamma m k_0 \log(n/(m k_0))$. Since $k_0 + 1 \leq 2k_0$, obtain that $k_0 \geq n_0/(2m\gamma)$. Hence, for any $k_0 \geq 0$

$$\log |\mathcal{P}(m, n, n_0, \gamma)| \geq \frac{n_0}{2} \log(\gamma n/n_0). \quad (7.25)$$

For every even value of l , apply Lemma 3 with $r = n_0/40$ obtaining that there exists a subset $\mathcal{S}_l(m, n, n_0, \gamma)$ of the set $\mathcal{P}(m, n, n_0, \gamma)$ such that, for any $\mathbf{C}^{(l)}, \mathbf{C}'^{(l)} \in \mathcal{S}_l(m, n, n_0, \gamma)$, one has

$$\|\mathbf{C}^{(l)} - \mathbf{C}'^{(l)}\|_H \geq n_0/40, \quad l = 2k, \quad k = 1, \dots, [L/2]. \quad (7.26)$$

By (7.25) and Lemma 3, for every even l , one has

$$\log |\mathcal{S}_l(m, n, n_0, \gamma)| \geq \frac{n_0}{2} \log \left(\frac{\gamma n}{n_0} \right) - \frac{n_0}{40} \log \left(\frac{80 n e m \gamma^2}{n_0} \right) \geq \frac{n_0}{40} \log \left(\frac{n e m}{n_0} \right)$$

if $\gamma n/n_0 \geq m^{1/9}(80e^2)^{1/18} \leq 0.75 m^{1/10}$. For odd values of l , let $\mathcal{S}_l(m, n, n_0, \gamma)$ contain just one clustering matrix corresponding to the uniform sequential clustering. Now, consider the set

$$\mathcal{S}(m, n, n_0, \gamma, L) = \prod_{l=1}^L \mathcal{S}_l(m, n, n_0, \gamma) \text{ with } \log |\mathcal{S}(m, n, n_0, \gamma, L)| \geq \frac{(L-1)n_0}{80} \log \left(\frac{n e m}{n_0} \right). \quad (7.27)$$

Let $\mathbf{C} = (\mathbf{C}^{(1)}, \mathbf{C}^{(2)}, \dots, \mathbf{C}^{(L)})$ and $\mathbf{C}' = (\mathbf{C}'^{(1)}, \mathbf{C}'^{(2)}, \dots, \mathbf{C}'^{(L)})$ be two sets of clustering matrices with $\mathbf{C}^{(l)}, \mathbf{C}'^{(l)} \in \mathcal{S}_l(m, n, n_0, \gamma)$ and let $z = (z_1, \dots, z_L)$ and $z' = (z'_1, \dots, z'_L)$ be the corresponding clustering functions. Let $\mathbf{\Lambda}$ and $\mathbf{\Lambda}'$ be the tensors of probabilities corresponding to sets of clustering

matrices $\mathbf{C}, \mathbf{C}' \in \mathcal{S}(m, n, n_0, \gamma, L)$. Then, similarly to the previous case, using (7.17), derive

$$\begin{aligned} \|\mathbf{\Lambda} - \mathbf{\Lambda}'\|^2 &= 2 \sum_{l=1}^{\lfloor L/2 \rfloor} \sum_{i=1}^{n(1-\gamma)} \sum_{j=n(1-\gamma)+1}^n (\mathbf{F}_{z_{2l}(i), z_{2l}(j)} - \mathbf{F}_{z'_{2l}(i), z'_{2l}(j)})^2 \\ &= \frac{2n}{m} \sum_{l=1}^{\lfloor L/2 \rfloor} \sum_{k=1}^{m(1-\gamma)} \sum_{j=n(1-\gamma)+1}^n (\mathbf{F}_{k, z_{2l}(j)} - \mathbf{F}_{k, z'_{2l}(j)})^2 \\ &= \frac{2n}{m} \sum_{l=1}^{\lfloor L/2 \rfloor} \sum_{j=n(1-\gamma)+1}^n \|\mathbf{F}_{*, z_{2l}(j)} - \mathbf{F}_{*, z'_{2l}(j)}\|^2 \geq \frac{n\rho^2}{8} \sum_{l=1}^{\lfloor L/2 \rfloor} \|\mathbf{C}^{(2l)} - \mathbf{C}'^{(2l)}\|_H, \end{aligned}$$

so that by (7.26),

$$\|\mathbf{\Lambda} - \mathbf{\Lambda}'\|^2 \geq (L-1)nn_0\rho^2/1280. \quad (7.28)$$

Again, similarly to the previous case, if $\rho \leq 1/4$, then by Proposition 4.2 of Gao *et al.* (2015), obtain that the Kullback divergence is bounded above

$$K(\mathbb{P}_{\mathbf{\Lambda}}, \mathbb{P}_{\mathbf{\Lambda}'}) \leq 8 \|\mathbf{\Lambda} - \mathbf{\Lambda}'\|^2 \leq 8Lnn_0\rho^2. \quad (7.29)$$

Set $\rho^2 = C_\rho n^{-1} \log(nem/n_0)$ where C_ρ is an absolute constant and apply Theorem 2.5 of Tsybakov (2009). Observe that if C_ρ is small enough, then, due to (7.27) and (7.29), conditions of this theorem are satisfied, hence,

$$\inf_{\widehat{\mathbf{\Lambda}}} \sup_{\substack{\mathbf{G} \in \mathcal{G}_{m,L,s} \\ \mathbf{C} \in \mathcal{S}(m, n, n_0, \gamma, L)}} \mathbb{P}_{\mathbf{\Lambda}} \left\{ \frac{\|\widehat{\mathbf{\Lambda}} - \mathbf{\Lambda}^*\|^2}{n^2 L} \geq C_\gamma \frac{n_0}{n^2} \log \left(\frac{nem}{n_0} \right) \right\} \geq 1/4. \quad (7.30)$$

The nonparametric estimation error. Consider sequential uniform clustering with n/m nodes in each group and group memberships remaining the same for all $l = 1, \dots, L$. Let $\mathbf{Q} \in \mathbb{R}^{M \times L}$ be the matrix with columns $\mathbf{q}^{(l)}$, $l = 1, \dots, L$, defined in Section 2.2. Denote $\mathbf{V} = \mathbf{Q}\mathbf{H}^T \in \mathbb{R}^{M \times L}$ and recall that for $\mathbf{G} \in \mathcal{G}_{m,L,s}$, by (3.1) and (3.2), matrix \mathbf{V} should have at most s nonzero entries.

Let $k_0 = \min(s/2, M)$. Choose k_0 rows among M rows of matrix \mathbf{V} and denote this set by \mathcal{X} . If $k_0 = M$, set $\mathcal{X} = \{1, \dots, M\}$. We have already distributed k_0 non-zero entries and have $s - k_0$ entries left. We distribute those entries into the k_0 rows $\mathbf{V}_{k,*}$ where $k \in \mathcal{X}$. Let

$$s_0 = \lceil (s - k_0)/k_0 \rceil = \lceil s/k_0 \rceil - 1 \quad \text{with} \quad s/2 \leq k_0 s_0 < s, \quad (7.31)$$

where $\lceil s/k_0 \rceil$ is the largest integer no larger than s/k_0 . Consider a set of binary vectors $\boldsymbol{\omega} \in \{0, 1\}^L$ with exactly s_0 ones in each vector. By Lemma 4.10 of Massart (2007), there exists a subset \mathcal{T} of those vectors such that for any $\boldsymbol{\omega}, \boldsymbol{\omega}' \in \mathcal{T}$, one has

$$\|\boldsymbol{\omega} - \boldsymbol{\omega}'\|_H \geq s_0/2 \quad \text{and} \quad \log |\mathcal{T}| \geq 0.233 s_0 \log(L/s_0).$$

Denote $\tilde{\mathcal{T}} = \prod_{k \in \mathcal{X}} \mathcal{T}_k$, where \mathcal{T}_k is a copy of the set \mathcal{T} corresponding to row k of matrix \mathbf{V} . For $\boldsymbol{\omega}^{(k)} \in \mathcal{T}_k$, set

$$\mathbf{V}_{k,*} = (\sqrt{L}/2, \dots, 0) + \rho m/n \boldsymbol{\omega}^{(k)}, \quad \text{if } k \in \mathcal{X}, \quad \mathbf{V}_{k,*} = 0 \quad \text{if } k \notin \mathcal{X}. \quad (7.32)$$

It is easy to see that matrix \mathbf{V} has at most s nonzero entries as required.

Let \mathbf{V} and \mathbf{V}' be matrices corresponding to sequences $\boldsymbol{\omega}^{(k)}$ and $\boldsymbol{\omega}'^{(k)}$ in \mathcal{T}_k , $k \in \mathcal{X}$. Let $\mathbf{\Lambda}$ and $\mathbf{\Lambda}'$ be the tensors corresponding to \mathbf{V} and \mathbf{V}' . Then, due to uniform clustering and (7.31), (7.32) implies that

$$\begin{aligned}\|\mathbf{\Lambda} - \mathbf{\Lambda}'\|^2 &\geq \left(\frac{n}{m}\right)^2 \|\mathbf{V} - \mathbf{V}'\|_F^2 \geq \left(\frac{n}{m}\right)^2 \sum_{k \in \mathcal{X}} \|\mathbf{V}_{k,*} - \mathbf{V}'_{k,*}\|^2 \geq \frac{\rho^2 k_0 s_0}{2} \geq \frac{\rho^2 s}{4}; \\ \|\mathbf{\Lambda} - \mathbf{\Lambda}'\|^2 &\leq 4 \left(\frac{m}{n}\right)^2 \rho^2 \left(\frac{n}{m}\right)^2 k_0 s_0 \leq 4\rho^2 s.\end{aligned}$$

Set $\rho = C_\rho \log(L/s_0)$. It is easy to check that, due to assumptions (4.1) and (4.2), one has $\mathbf{Q}_{ij} \in [1/4, 3/4]$ for any i and j . Hence, by Lemma 4.2 of Gao *et al.* (2015), obtain $K(\mathbb{P}_{\mathbf{\Lambda}}, \mathbb{P}_{\mathbf{\Lambda}'}) \leq 8 \|\mathbf{\Lambda} - \mathbf{\Lambda}'\|^2 \leq 32\rho^2 s$. If C_ρ is small enough, conditions of Theorem 2.5 of Tsybakov (2009) hold. Finally, observe that if $s < 2M$, then $k_0 = s/2$, $s_0 = 1$ and $L/s_0 = L = Lm^2/m^2 \geq \gamma Lm^2/s$. If $s \geq 2M$, then $k_0 = M$, $s_0 \leq s/M$ and $L/s_0 \geq LM/s \geq Lm^2/(2s) \geq \gamma Lm^2/s$. Since for some constant C_γ , one has $\log(\gamma Lm^2/s) \geq C_\gamma \log(Lm^2/s)$, one has

$$\inf_{\mathbf{\Lambda}} \sup_{\substack{\mathbf{G} \in \mathcal{G}_{m,L,s} \\ \mathbf{C} \in \mathcal{S}(m,n,n_0,\gamma,L)}} \mathbb{P}_{\mathbf{\Lambda}} \left\{ \frac{\|\hat{\mathbf{\Lambda}} - \mathbf{\Lambda}^*\|^2}{n^2 L} \geq C_\gamma \frac{s}{n^2 L} \log \left(\frac{Lm^2}{s} \right) \right\} \geq 1/4. \quad (7.33)$$

Finally, in order to obtain the lower bound in (4.3) observe that $\max(a, b, c) \leq a + b + c \leq 3 \max(a, b, c)$ for any $a, b, c \geq 0$ and combine (7.23), (7.30) and (7.33).

Proof of Theorem 4. We consider cases when $r = r_{n,L}$ and $r = r_0$ corresponding to piecewise constant and piecewise smooth graphon separately.

Piecewise constant graphon. Assume, without loss of generality, that $h = n/r$ is an integer. Consider a set up where the nodes are grouped into r classes and values of ζ_j 's are fixed:

$$\zeta_{kh+i} = \beta_k + i(\beta_{k+1} - \beta_k)/h, \quad k = 0, \dots, r-1, \quad i = 1, \dots, h.$$

Then, there are h nodes in each class. Let $\mathbf{G}_{k,k,l} = 0$ for any $k = 1, \dots, r$ and $l = 1, \dots, L$.

Consider an even L_0 such that $1 \leq L_0 \leq L/2$ and a set of vectors $\boldsymbol{\omega} \in \{0, 1\}^{L_0}$ with exactly $L_1 = L_0/2$ nonzero entries. By Lemma 4.10 of Massart (2007), there exists a subset \mathcal{T} of those vectors such that for any $\boldsymbol{\omega}, \boldsymbol{\omega}' \in \mathcal{T}$ one has

$$\|\boldsymbol{\omega} - \boldsymbol{\omega}'\|_H \geq L_0/4, \quad \log |\mathcal{T}| \geq 0.233(L_0/2) \log 2 \geq 0.08L_0. \quad (7.34)$$

Denote $\mathcal{K} = \{(k_1, k_2) : 1 \leq k_1 < k_2 \leq r\}$ and let \mathcal{T}_{k_1, k_2} be the copies of \mathcal{T} for $(k_1, k_2) \in \mathcal{K}$. Denote $\tilde{\mathcal{T}} = \prod_{(k_1, k_2) \in \mathcal{K}} \mathcal{T}_{k_1, k_2}$ and consider a set of functions $f^{(\tilde{\boldsymbol{\omega}})}$ with $\tilde{\boldsymbol{\omega}} \in \tilde{\mathcal{T}}$ such that, for $\beta_{k_1-1} < x \leq \beta_{k_1}$ and $\beta_{k_2-1} < y \leq \beta_{k_2}$ one has

$$\mathbf{v}_1^{(\tilde{\boldsymbol{\omega}})}(x, y) = \sqrt{L}/2; \quad \mathbf{v}_l^{(\tilde{\boldsymbol{\omega}})}(x, y) = \rho \boldsymbol{\omega}_{l-L_0}^{(k_1, k_2)}, \quad l = L_0 + 1, \dots, 2L_0, \quad (k_1, k_2) \in \mathcal{K},$$

where $\tilde{\boldsymbol{\omega}}$ is a binary matrix with elements $\boldsymbol{\omega}_l^{(k_1, k_2)}$, $l = 1, \dots, L_0$ and $(k_1, k_2) \in \mathcal{K}$. Then, Assumption (5.1) holds. In order (5.2) is satisfied, we set

$$\rho^2 \leq C_1 L_0^{-(2\nu_2+1)}, \quad C_1 \leq \min(K_2 2^{1-2\nu_2}, 1/8). \quad (7.35)$$

Denote by $\mathbf{\Lambda}$ and $\mathbf{\Lambda}'$ the probability tensors corresponding, respectively, to $\tilde{\omega}$ and $\tilde{\omega}'$ in $\tilde{\mathcal{T}}$. Then, due to (7.34) and symmetry,

$$\begin{aligned}\|\mathbf{\Lambda} - \mathbf{\Lambda}'\|^2 &\geq 2\rho^2 \sum_{k_1=1}^r \sum_{k_2=k_1+1}^r \|\omega^{(k_1, k_2)} - \omega'^{(k_1, k_2)}\|_H \left(\frac{n}{r}\right)^2 \geq \frac{\rho^2 n^2 L_0}{8}; \\ \|\mathbf{\Lambda} - \mathbf{\Lambda}'\|^2 &\leq \rho^2 r(r-1)(n/r)^2 L_0 \leq \rho^2 n^2 L_0.\end{aligned}$$

Note that one has $1/4 \leq f(x, y, t) \leq 3/4$ provided $\|\mathbf{H}^T \omega^{(k_1, k_2)}\|_\infty \leq 1/4$ for $(k_1, k_2) \in \mathcal{K}$. By Assumption (4.1), the latter is guaranteed by $\rho^2 L_0^2 / L \leq 1/4$, so that, due to $L \geq 2L_0$ and $\nu_2 \geq 1/2$, it is ensured by (7.35). Then, by Proposition 4.2 of Gao *et al.* (2015), one has

$$K(\mathbb{P}_{\mathbf{\Lambda}}, \mathbb{P}_{\mathbf{\Lambda}'}) \leq 8 \|\mathbf{\Lambda} - \mathbf{\Lambda}'\|^2 \leq 8\rho^2 n^2 L_0 \leq \log |\tilde{\mathcal{T}}|/8$$

provided

$$\rho^2 \leq C_2 (r/n)^2 \quad (7.36)$$

where C_2 is an absolute constant. Therefore, application of Theorem 2.5 of Tsybakov (2009) yields (5.6) with $\Delta(n, L) = C \rho^2 L_0 / L$ where C is an absolute constant.

Now, we set $C_3^2 = (C_2/C_1)2^{-(2\nu_2+1)}$ and consider two cases. If $n \leq C_3 r L^{\nu_2+1/2}$, choose L_0 such that $\rho^2 = C_1 L_0^{-(2\nu_2+1)} = C_2 (r/n)^2$ which holds when $L_0 = [(C_1/C_2)(n/r)^2]^{1/(2\nu_2+1)}$. It is easy to check that $L_0 \geq 2$ and that $L_0 \leq L/2$, so that

$$\Delta(n, L) = \frac{C}{L} \left[\left(\frac{r}{n} \right)^2 \right]^{\frac{2\nu_2}{2\nu_2+1}}.$$

If $n > C_3 r L^{\nu_2+1/2}$, choose $\rho^2 = C_2 (r/n)^2$ and set $L_0 = L/2$. Then, (7.35) holds and $\Delta(n, L) = C(r/n)^2$ which completes the proof of the lower bound when $r = r_{n,L}$.

Piecewise smooth graphon. Since $r = r_0$ is a fixed quantity, without loss of generality, we set $r = 1$. Let $\zeta_j = j/n, j = 1, \dots, n$, be fixed. Let h be a positive integer, $1 \leq h \leq n$, and denote $\delta = 1/h$. Consider a kernel function $F(x)$ such that $F(x)$ is $n_1 > \nu_1$ times continuously differentiable and for any $x, x' \in \mathbb{R}$ and some $C_F > 0$

$$\text{supp}(F) = (-1/2; 1/2), \quad |F(x) - F(x')| \leq C_F |x - x'|^{\nu_1}. \quad (7.37)$$

It follows from (7.37) that $|F(x)| \leq C_F$ for any x . Denote

$$\Psi_{k_1, k_2}(x, y) = h^{-\nu_1} F(h(x - u_{k_1})) F(h(y - u_{k_2})) \quad \text{where} \quad u_k = (k-1)\delta + \delta/2, \quad k = 1, \dots, h. \quad (7.38)$$

It is easy to see that $\Psi_{k_1, k_2}(x, y) = \Psi_{k_2, k_1}(y, x)$ and, for different pairs (k_1, k_2) , functions $\Psi_{k_1, k_2}(x, y)$ have disjoint supports. Similar to the case of the piecewise constant graphon, consider an even $L_0 \leq L/2$ and a set of vectors $\omega \in \{0, 1\}^{L_0}$ with exactly $L_0/2$ nonzero entries. By Lemma 4.10 of Massart (2007), there exists a subset \mathcal{T} of those vectors such that (7.34) holds for any $\omega, \omega' \in \mathcal{T}$. Let again \mathcal{T}_{k_1, k_2} be the copies of \mathcal{T} for $(k_1, k_2) \in \mathcal{K}$ and denote $\tilde{\mathcal{T}} = \prod_{(k_1, k_2) \in \mathcal{K}} \mathcal{T}_{k_1, k_2}$ where $\mathcal{K} = \{(k_1, k_2) : 1 \leq k_1 < k_2 \leq h\}$. Then, $\log |\tilde{\mathcal{T}}| \geq 0.04 L_0 h^2$. For any $\tilde{\omega} \in \tilde{\mathcal{T}}$ and $l = L_0 + 1, \dots, 2L_0$, define

$$\mathbf{v}_1(x, y) = \sqrt{L}/2; \quad \mathbf{v}_l(x, y) = \rho \sum_{k_1=1}^h \sum_{k_2=k_1+1}^h \omega_{l-L_0}^{(k_1, k_2)} [\Psi_{k_1, k_2}(x, y) + \Psi_{k_2, k_1}(x, y)]. \quad (7.39)$$

It is easy to see that $\mathbf{v}(x, y) = \mathbf{v}(y, x)$ for any $x, y \in [0, 1]$. Now we need to check that conditions (5.1) and (5.2) hold. Note that for any x, y, x', y' , due to (7.37), obtain

$$|\Psi_{k_1, k_2}(x, y) - \Psi_{k_1, k_2}(x', y')| \leq h^{-\nu_1} [|F(h(x - u_{k_1}) - F(h(x' - u_{k_1}))|F(h(y - u_{k_2}))| \\ + |F(h(y - u_{k_2}) - F(h(y' - u_{k_2}))|F(h(x' - u_{k_1}))|] \leq C_\psi [|x - x'| + |y - y'|]^{\nu_1},$$

where constant C_ψ depends only on C_F and ν_1 . Since functions $\Psi_{k_1, k_2}(x, y)$ have disjoint supports for different pairs (k_1, k_2) , the latter implies that (5.1) holds if $\rho \leq K_1/(4C_\psi)$. Also, it is easy to check that, by (7.37), one has $[\mathbf{v}_l(x, y)]^2 \leq C_v \rho^2 h^{-2\nu_1}$ where C_v depends only on C_F and ν_1 . Therefore,

$$\sum_{l=L_0+1}^{2L_0} l^{2\nu_2} \mathbf{v}_l^2(x, y) \leq C_v^2 \rho^2 h^{-2\nu_1} L_0^{2\nu_2+1}.$$

Hence, both assumptions, (5.1) and (5.2) are valid provided

$$\rho \leq \min \left(K_1/(4C_\psi), K_2/C_v h^{\nu_1} L_0^{-(\nu_2+1/2)} \right). \quad (7.40)$$

Now, note that, by (4.1) and (7.39), since $\Psi_{k_1, k_2}(x, y)$ have disjoint supports, we derive that $f(x, y, t) \in [1/4; 3/4]$ provided

$$\rho \leq \sqrt{L}/(4C_v^2 L_0). \quad (7.41)$$

Then, by Proposition 4.2 of Gao *et al.* (2015), one has

$$K(\mathbb{P}_{\mathbf{\Lambda}}, \mathbb{P}_{\mathbf{\Lambda}'}) \leq 4\rho^2 h^{-2\nu_1} L_0 \left[\sum_{i=1}^n \sum_{k=1}^h F^2(h(i/n - u_k)) \right]^2.$$

Here,

$$\sum_{i=1}^n \sum_{k=1}^h F^2(h(i/n - u_k)) = \sum_{k=1}^h \sum_{u_k - \delta/2 < i/n \leq u_k + \delta/2} F^2(h(i/n - u_k)) \\ \approx n \sum_{k=1}^h \int_{u_k - \delta/2}^{u_k + \delta/2} F^2(h(i/n - u_k)) dx = n \int_{-1/2}^{1/2} F^2(z) dz, \quad (7.42)$$

so that $K(\mathbb{P}_{\mathbf{\Lambda}}, \mathbb{P}_{\mathbf{\Lambda}'}) \leq 4 \|F\|_2^4 \rho^2 h^{-2\nu_1} n^2 L_0$ where $\|F\|_2$ is the L^2 -norm of F .

Denote by $\mathbf{\Lambda}$ and $\mathbf{\Lambda}'$ the probability tensors corresponding to $\tilde{\omega}$ and $\tilde{\omega}'$ in $\tilde{\mathcal{T}}$, respectively. Then, due to (7.34) and (7.42), obtain

$$\|\mathbf{\Lambda} - \mathbf{\Lambda}'\|^2 \geq 2\rho^2 \sum_{l=L_0+1}^{2L_0} \sum_{k_1=1}^r \sum_{k_2=k_1+1}^r \sum_{i=1}^n \sum_{j=1}^n [\omega_l^{(k_1, k_2)} - \omega_l'^{(k_1, k_2)}]^2 [\Psi_{k_1, k_2}(i/n, j/n)]^2 \\ \geq \frac{\rho^2 h^{-2\nu_1} L_0}{4} \left[\sum_{i=1}^n \sum_{k=1}^h F^2(h(i/n - u_k)) \right]^2 \geq \frac{\rho^2 h^{-2\nu_1} L_0 n^2}{8} \|F\|_2^4.$$

Application of Theorem 2.5 of Tsybakov (2009) yields that (5.6) holds with $\Delta(n, L) = C \rho^2 h^{-2\nu_1} L_0/L$ provided $4\|F\|_2^4 \rho^2 h^{-2\nu_1} n^2 L_0 \leq 0.08/8 L_0 h^2/2$, which requires the following restriction on ρ :

$$\rho \leq h^{\nu_1+1} n^{-1} / (20\sqrt{2} \|F\|_2^2). \quad (7.43)$$

Set

$$h = n^{\frac{1}{\nu_1+1}}, \quad L_0 = \min \left(n^{\frac{2\nu_1}{(\nu_1+1)(2\nu_2+1)}}, \frac{L}{2} \right), \quad n_L = \left(\frac{L}{2} \right)^{\frac{(\nu_1+1)(2\nu_2+1)}{2\nu_1}}. \quad (7.44)$$

and consider two cases.

If $n \leq n_L$, then L_0 is given by the first expression in (7.44) and inequalities (7.40), (7.41) and (7.43) hold if ρ^2 is a small enough absolute constant that depends on ν_1, ν_2, K_1 and K_2 only. In this case, $\Delta(n, L) = C L^{-1} n^{\frac{-4\nu_1\nu_2}{(\nu_1+1)(2\nu_2+1)}}$ and (5.7) is valid.

If $n > n_L$, then $L_0 = L/2$ and again all inequalities (7.40), (7.41) and (7.43) hold if ρ^2 is a small enough absolute constant that depends on ν_1, ν_2, K_1 and K_2 only. In this case, $\Delta(n, L) = C n^{\frac{-2\nu_1}{\nu_1+1}}$ which completes the proof.

7.3 Proofs of supplementary statements

Lemma 1 *Let a_i be independent Bernoulli(θ_i) variables and consider vectors \mathbf{a} and $\boldsymbol{\theta}$ with components a_i and θ_i , respectively. Then, for any vector \mathbf{z} and any positive t and α one has*

$$\mathbb{P} \left(|\mathbf{z}^T(\mathbf{a} - \boldsymbol{\theta})| > \frac{\alpha \|\mathbf{z}\|^2}{2} + \frac{t}{\alpha} \right) \leq 2e^{-t}, \quad \mathbb{E} [\exp(\mathbf{z}^T(\mathbf{a} - \boldsymbol{\theta}))] \leq \exp(\|\mathbf{z}\|^2/8). \quad (7.45)$$

Proof. Validity of the Lemma follows from Hoeffding inequality (see, e.g., Massart (2007)).

Lemma 2 *Let a_i be independent Bernoulli(θ_i) variables and consider vectors \mathbf{a} and $\boldsymbol{\theta}$ with components a_i and θ_i , respectively. Then, for any matrix \mathbf{A} and any positive t one has*

$$\mathbb{P} \left(\|\mathbf{A}(\mathbf{a} - \boldsymbol{\theta})\|^2 > \frac{\|\mathbf{A}\|_F^2}{2} + \frac{3t \|\mathbf{A}\|^2}{4} \right) \leq e^{-t}. \quad (7.46)$$

Proof. Denote $\boldsymbol{\xi} = \mathbf{a} - \boldsymbol{\theta}$, $\boldsymbol{\Sigma} = \mathbf{A}^T \mathbf{A}$ and use Theorem 2.1 of Hsu *et al.* (2012). Due to (7.45), conditions of the theorem hold with $\sigma = 1/2$, hence, for any $t > 0$ one has

$$\mathbb{P} \left[\|\mathbf{A}\boldsymbol{\xi}\|^2 > \frac{1}{4} \left(\text{Tr}(\boldsymbol{\Sigma}) + 2\sqrt{t \text{Tr}(\boldsymbol{\Sigma}^2)} + 2\|\boldsymbol{\Sigma}\|t \right) \right] \leq e^{-t}.$$

Note that $\text{Tr}(\boldsymbol{\Sigma}^2) \leq \|\boldsymbol{\Sigma}\| \text{Tr}(\boldsymbol{\Sigma})$ and $2\sqrt{xy} \leq x + y$ for any positive x and y . In order to complete the proof, recall that $\|\boldsymbol{\Sigma}\| = \|\mathbf{A}\|^2$ and $\text{Tr}(\boldsymbol{\Sigma}) = \|\mathbf{A}\|_F^2$.

Lemma 3 (The packing lemma). *Let $\mathcal{Z}(m, n) \subseteq \mathcal{M}(m, n)$ be a collection of clustering matrices. Then, there exists a subset $S_{n,m}(r) \subset \mathcal{Z}(m, n)$ such that for $\mathbf{C}_1, \mathbf{C}_2 \in \mathcal{Z}(m, n)$ one has $\|\mathbf{C}_1 - \mathbf{C}_2\|_H = \|\mathbf{C}_1 - \mathbf{C}_2\|_F^2 \geq r$ and $\log |S_{n,m}(r)| \geq \log |\mathcal{Z}(m, n)| - r \log(nem/r)$.*

Proof. For any clustering matrix \mathbf{C} define the r -neighborhood of \mathbf{C} as

$$\mathcal{B}(\mathbf{C}, r) = \left\{ \tilde{\mathbf{C}} \in \mathcal{Z}(m, n) : \|\tilde{\mathbf{C}} - \mathbf{C}\|_H \leq r \right\}.$$

Let $\tilde{S}_{n,m}(r)$ be the covering set of $\mathcal{Z}(m, n)$ and $|\tilde{S}_{n,m}(r)|$ be the covering number, the smallest number of closed balls of radius r whose union covers $\mathcal{Z}(m, n)$. Let $|S_{n,m}(r)|$ be the packing number

of $\mathcal{Z}(m, n)$, the largest number of points with the distance at least r between them. It is known (see Pollard (1990), page 10) that

$$|\tilde{S}_{n,m}(r)| \leq |S_{n,m}(r)| \leq |\tilde{S}_{n,m}(r/2)| \quad (7.47)$$

Note that $|\mathcal{Z}(m, n)| \leq \sum_{\mathbf{C} \in \tilde{S}_{n,m}(r)} |\mathcal{B}(\mathbf{C}, r)| \leq |\tilde{S}_{n,m}(r)| \max_{\mathbf{C} \in \tilde{S}_{n,m}(r)} |\mathcal{B}(\mathbf{C}, r)|$, so that

$$|\tilde{S}_{n,m}(r)| \geq |\mathcal{Z}(m, n)| \left/ \max_{\mathbf{C} \in \tilde{S}_{n,m}(r)} |\mathcal{B}(\mathbf{C}, r)| \right. \quad (7.48)$$

and, also,

$$|\mathcal{B}(\mathbf{C}, r)| \leq \binom{n}{r} m^r \leq \left(\frac{ne}{r}\right)^r m^r = \left(\frac{nem}{r}\right)^r. \quad (7.49)$$

Now, combining (7.47) – (7.49), obtain $\log |S_{n,m}(r)| \geq \log |\tilde{S}_{n,m}(r)| \geq \log |\mathcal{Z}(m, n)| - r \log(nem/r)$ which completes the proof.

Lemma 4 *Let γm and n/m be positive integers. Then, for $n_0 \geq 1$ and $\gamma m \geq 2$, one has*

$$\log \{(\gamma n)!\} - \gamma m \log \{(n/m)!\} \geq \frac{\gamma n}{4} \log(\gamma m); \quad (7.50)$$

$$\log \left\{ \binom{m\gamma}{n_0} \left(\frac{n}{m}\right)^{n_0} (n_0 - 1)! \right\} \geq n_0 \log \left(\frac{\gamma n}{n_0}\right) + \frac{n_0}{2} \log \left(\frac{n_0}{2}\right); \quad (7.51)$$

$$6\gamma n \log(\gamma m) - \gamma n \log(32m\gamma e) \geq 0. \quad (7.52)$$

Proof. Note that due to (7.18), one has

$$A_1 = \log \{(\gamma n)!\} - \gamma m \log \{(n/m)!\} \approx \gamma n \log(\gamma m) - \frac{\gamma m}{2} \log \left(\frac{2\pi n}{m}\right) + \frac{1}{2} \log(2\pi\gamma n).$$

Consider a function

$$F(x) = \frac{3\gamma n}{4} \log x - \frac{x}{2} \log \left(\frac{2\pi\gamma n}{x}\right) + \frac{1}{2} \log(2\pi\gamma n).$$

It is easy to check that $F(1) = 0$ and that $F'(x) = 0.75\gamma n/x - 0.5 \log(2\pi\gamma n/(xe))$. Replacing $2\pi\gamma n/(xe)$ in $F'(x)$ by z and noting that the inequality $3e/(4\pi)z > \log z$ is true for any $z > 0$, we confirm that $F'(x) > 0$. so that $F(x) > 0$ for any $x \geq 1$. Finally, in order to prove (7.50), observe that $A_1 = F(\gamma m) + \gamma n \log(\gamma m)/4$.

For the sake of proving (7.51), note that for every $n_0 \geq 1$ one has $\log[(n_0-1)!] \geq 0.5 n_0 \log(n_0/2)$ and apply the first inequality in (7.18).

The validity of inequality (7.52) follows from $\gamma m \geq 2$ and the fact that $\log(64e) < 6$.

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